I $AB \cdot B^{-1}A^{-1} = AI_2A^{-1} = AA^{-1} = I_2$

$B^{-1}A^{-1}AB = B^{-1}I_2B = B^{-1}B = I_2.$

Thus AB is regular with its inverse

$(AB)^{-1} = B^{-1}A^{-1}.$

$A^{-1} \cdot A = I_2$ and $A \cdot A^{-1} = I_2$

Thus A^{-1} is regular with its inverse

$(A^{-1})^{-1} = A.$

II. 1st. solution take $\vec{x} = \vec{a}$. Then

$\|\vec{a}\|^2 = (\vec{a}, \vec{a}) = 0$

Thus $\vec{a} = \vec{0}$.

2nd solution

set $\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ by components. Then

$0 = (\vec{a}, \vec{e}_j) = a_j$. $(j = 1, \cdots, n)$. This means $\vec{a} = \vec{0}$.

III. $A = (\vec{a}_1\ \vec{a}_2)$. Then

$\vec{0} = A\vec{e}_1 = \vec{a}_1$, $\vec{0} = A\vec{e}_2 = \vec{a}_2$. Thus

$A = (\vec{0}\ \vec{0}) = O_2.$

(2

IV (1) $A^{-1} = -\frac{1}{3}\begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}$ (2) $A^{-1} = \frac{1}{6}\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$

(3) $A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ (4) $A^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$

V. (i)
$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\left(\begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\right)$$
$$= \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 3 & 5 \end{pmatrix}$$

(ii)
$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\left(\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\right)$$
$$= \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 6 & 2 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 16 & 10 \\ 16 & 10 \end{pmatrix}$$

(iii)
$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 6 \\ 5 & 10 \end{pmatrix} + \begin{pmatrix} 7 & 5 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 11 \\ 12 & 15 \end{pmatrix}$$

VI (i) $A^2 = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^3 = A^2 A = I_2 A = A$

$A^4 = A^3 \cdot A = A A = I_2$

$$A^n = \begin{cases} I_n & n : \text{even} \\ A & n : \text{odd} \end{cases}$$

(ii) $A^2 = \begin{pmatrix} 3 & -9 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 3 & -9 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A^3 = A^2 \cdot A = O_2 \cdot A = O_2$

$A^4 = A^3 \cdot A = O_2 \cdot A = O_2$

$$A^n = \begin{cases} A & n = 1 \\ O_2 & n \geq 2 \end{cases}$$

(iii) $A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$A^3 = A^2 \cdot A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

Assume $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

Then $A^{n+1} = A^n \cdot A = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}$

Thus we have for any $n \in \mathbb{N}$

$$A^n = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}$$

(iii) $A^2 = \begin{pmatrix} 1 & 6 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

$$A^3 = A^2 A = I_2 A = A$$

$$A^4 = A^3 \cdot A = A \cdot A = I_2$$

$$A^n = \begin{cases} A & n : \text{odd} \\ I_2 & n : \text{even}. \end{cases}$$

VII. (i) Since $\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5 \neq 0$, $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ is regular. Multiply $\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}^{-1}$ to the both sides from the left hand side to get

$$X = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -7 & -8 \\ 8 & 10 \end{pmatrix}$$

(ii) Since $\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 18 - 20 = -2 \neq 0$, $\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$ is regular. Multiply $\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}^{-1}$ to the both sides from the right to get

$$X = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\left(-\frac{1}{2}\right)\begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -4 & 2 \\ -17 & 5 \end{pmatrix}$$

V III $AX + B = C \iff AX = -B + C$.

Since $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = -1 \neq 0$, A is regular.

Multiply A^{-1} to the both sides to get

$$X = A^{-1}(-B + C)$$

$$= -\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}\left(-\begin{pmatrix} -4 & 4 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}\right)$$

$$= -\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 6 & -4 \\ 4 & 1 \end{pmatrix}$$

$$= -\begin{pmatrix} 22 & -22 \\ -14 & 13 \end{pmatrix} = \begin{pmatrix} -22 & 22 \\ 14 & -13 \end{pmatrix}$$

IX (i) By C-H we get

$$(\#) \quad A^2 + 3A - 10 I_2 = O_2$$

The eigen-polynomial of A is calculated and factorized as

$$\Phi_A(\lambda) = \lambda^2 + 3\lambda - 10 = (\lambda+5)(\lambda-2).$$

With this factorization in mind, we reformulate (#) by

$$A^2 - ((-5)+2)A + (-5)\cdot 2\, I_2 = O_2.$$

to get

$$\begin{cases} A^2 + 5A = 2(A + 5I_2) \\ A^2 - 2A = (-5)(A - 2I_2), \end{cases}$$

We consider this as

$$\begin{cases} A(A+5I_2) = 2(A+5I_2) \\ A(A-2I_2) = (-5)(A-2I_2) \end{cases}$$

to get

$$\begin{cases} A^n(A+5I_2) = 2^n(A+5I_2) \\ A^n(A-2I_2) = (-5)^n(A-2I_2) \end{cases}$$

It follows for this that

$$\begin{cases} A^{n+1} + 5A^n = 2^n(A+5I_2) \quad \cdots (1) \\ A^{n+1} - 2A^n = (-5)^n(A-2I_2) \quad \cdots (2) \end{cases}$$

Finally (1) − (2) leads us to

$$7A^n = 2^n(A+5I_2) - (-5)^n(A-2I_2)$$

Accordingly we have shown that

$$A^n = \frac{1}{7}2^n(A+5I_2) - \frac{1}{7}(-5)^n(A-2I_2)$$

(ii) By C-H we get

$$(\#) \quad A^2 + 3A + 2I_2 = O_2$$

The eigen-polynomial of A is calculated and factorized as

$$\Phi_A(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1).$$

with this factorization in mind, we reformulate (#) by

$$A^2 - ((-1)+(-2))A + (-1)(-2)I_2 = O_2.$$

to get

$$\begin{cases} A(A+I_2) = (-2)(A+I_2) & \cdots (1) \\ A(A+2I_2) = (-1)(A+2I_2) & \cdots (2) \end{cases}$$

By using (1) an (2) repeatedly we get

$$\begin{cases} A^n(A+I_2) = (-2)^n(A+I_2) & \cdots (3) \\ A^n(A+2I_2) = (-1)^n(A+2I_2) & \cdots (4) \end{cases}$$

(4) $-$ (3) leads us to

$$A^n = (-1)^n(A+2I_2) - (-2)^n(A+I_2)$$

(iii) By C-H we get

$$(\#) \quad A^2 - 2A - 35 I_2 = O_2 .$$

The eigen-polynomials of A is calculated and factorized by

$$\Phi_A(\lambda) = \lambda^2 - 2\lambda - 35 = (\lambda - 7)(\lambda + 5).$$

With this in mind $(\#)$ is rewritten by

$$A^2 - (7 + (-5)) A + 7(-5) I_2 = O .$$

It follows from this that

$$\begin{cases} A(A - 7I_2) = (-5)(A - 7I_2) & \cdots (1) \\ A(A + 5I_2) = 7(A + 5I_2) & \cdots (2) \end{cases}$$

We apply (1) and (2) repeatedly to get

$$\begin{cases} A^n (A - 7I_2) = (-5)^n (A - 7I_2) & \cdots (3) \\ A^n (A + 5I_2) = 7^n (A + 5I_2) & \cdots (4) \end{cases}$$

(4) − (3) leads us to

$$-12 A^n = 7^n (A + 5I_2) - (-5)^n (A - 7I_2).$$

Accordingly we have shown that

$$A^n = \frac{1}{12}(-5)^n (A - 7I_2) - \frac{1}{12} 7^n (A + 5I_2)$$

XI (i) As is calculated in Exercise IX, the eigen-polynomial of A is given by

$$\Phi_A(\lambda) = \lambda^2 + 3\lambda - 10 = (\lambda+5)(\lambda-2).$$

Thus

$$\Phi_A(\lambda) = 0 \iff \lambda = -5 \text{ or } \lambda = 2$$

In case $\lambda = -$

$$(\#)_1 \quad (-5I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} -6 & -2 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff -3x + y = 0$$

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution to $(\#)_1$, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -3x \end{pmatrix} = x\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

In case $\lambda = 2$

$$(\#)_2 \quad (2I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff x - 2y = 0$$

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution to $(\#)_2$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(ii) The eigenpolynomial of A is calculated by

$$\Phi_A(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2).$$

Thus

$$\Phi_A(\lambda) = 0 \iff \lambda = -1 \text{ or } \lambda = -2.$$

In case $\lambda = -1$

$$(\#)_1 \quad (-I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff 3x + 2y = 0.$$

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution to $(\#)_1$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}y \\ y \end{pmatrix} = y\begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}.$$

In case $\lambda = -2$

$$(\#)_2 \quad (-2I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} 2 & 2 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\iff x + y = 0$$

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution to $(\#)_2$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(iii) The eigen-polynomials of A is calculated and factorized by

$$\Phi_A(\lambda) = \lambda^2 - 2\lambda - 35$$
$$= (\lambda - 7)(\lambda + 5).$$

Thus

$$\Phi_A(\lambda) = 0 \iff \lambda = -5 \text{ or } \lambda = 7.$$

In case $\lambda = -5$

$$(\#)_1 \quad (-5I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} -6 & -12 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$
$$\iff x + 2y = 0$$

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution to $(\#)_1$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

In case $\lambda = 7$

$$(\#)_2 \quad (7I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} 6 & -12 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$
$$\iff x - 2y = 0$$

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution to $(\#)_2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$