Special Matrices

O_2 Zero Matrix

$$O_2 = (\vec{0}\ \vec{0}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is called the zero matrix. It satisfies the identity

$$O_2 X = X O_2 = O_2$$

$O_2 X = O_2$ follows from $O_2 X = O_2 (\vec{x}_1\ \vec{x}_2) = (O_2 \vec{x}_1\ O_2 \vec{x}_2)$

$$O_2 \vec{x} = (\vec{0}\ \vec{0}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \vec{0} + x_2 \vec{0} = \vec{0} \qquad = (\vec{0}\ \vec{0})$$

and $X O_2 = O_2$ from $X O_2 = X (\vec{0}\ \vec{0}) = (X\vec{0}\ X\vec{0})$

$$X \vec{0} = (\vec{x}_1\ \vec{x}_2) \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\vec{x}_1 + 0\vec{x}_2 = \vec{0} \qquad = (\vec{0}\ \vec{0})$$

Special Matrices 2

I_2 Identity Matrix

$$I_2 = (\vec{e}_1\ \vec{e}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is called the **Identity Matrix**. It enjoys for a 2×2 matrix X, the identity

$$X I_2 = I_2 X = X$$

It follows from the identities

$$(\vec{a}\ \vec{b})\vec{e}_1 = 1\vec{a} + 0\vec{b} = \vec{a}, \quad (\vec{a}\ \vec{b})\vec{e}_2 = 0\vec{a} + 1\vec{b} = \vec{b}$$

that $= ((\vec{x}_1, \vec{x}_2)\vec{e}_1\ \ (\vec{x}_1, \vec{x}_2)\vec{e}_2)$

$$X I_2 = (\vec{x}_1\ \vec{x}_2)(\vec{e}_1\ \vec{e}_2) = (\vec{x}_1\ \vec{x}_2) = X$$

Moreover $I_2 X = X$ from

$$(\vec{e}_1\ \vec{e}_2) \begin{pmatrix} x \\ y \end{pmatrix} = x\vec{e}_1 + y\vec{e}_2 = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Inverse matrix

We have the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

Cofactor Matrix

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its cofactor matrix is defeined by

$|A| = ad - bc$

$$\tilde{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Then we have the identity

$$A\tilde{A} = \tilde{A}A = |A| \cdot I_2$$

Inverse Matrix 2

Assume that $|A| \neq 0$. Then multiply by $\frac{1}{|A|}$ and get

$$A \cdot \frac{1}{|A|}\tilde{A} = \frac{1}{|A|}\tilde{A} \cdot A = I_2$$

Inverse Matrix

In case $|A| \neq 0$, the **Inverse Matrix** of A is defined by

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^{-1} = \frac{1}{|A|}\tilde{A} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Regularity of Matrices, Uniqueness of Inverse

Regularity of Matrices

A 2×2 matrix A is called **regular** if there exists another 2×2 matrix X satisfying

$$AX = XA = I_2$$

In this situation X is called the **inverse** of A.

If $|A| \neq 0$, the A is regular.

- **(i)** If $|A| \neq 0$, A is regular.
- **(ii)** **(Uniqueness of the inverse)** Assume

$$AX = XA = I_2, \quad AY = YA = I_2$$

Then $X = Y$. In fact, from $AX = I_2$ multiplied by Y from the left follows

$$Y(AX) = YI_2 = Y$$

On the other hand, $Y(AX) = (YA)X = I_2X = X$.
Accordingly $X = Y$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

In case $|A| = 0$

Theorem A

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a 2×2 matrix with $|A| = 0$. Then there exists $\vec{v} \neq \vec{0}$ satisfying $A\vec{v} = \vec{0}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d \\ -c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$d = 0$ and $c = 0$.

Not

- **(i)** In case $\boxed{d \neq 0 \text{ or } c \neq 0,}$ $\begin{pmatrix} d \\ -c \end{pmatrix} \neq \vec{0}$ AND $A\begin{pmatrix} d \\ -c \end{pmatrix} = \vec{0}$.
- **(ii)** In case $b \neq 0$ or $c \neq 0$, $\begin{pmatrix} -b \\ a \end{pmatrix} \neq \vec{0}$ AND $A\begin{pmatrix} -b \\ a \end{pmatrix} = \vec{0}$.
- **(iii)** Not (i) AND Not (ii). Then $A = O_2$.

$\vec{v} \in \mathbb{R}^2$, $\neq \vec{0}$

$\Rightarrow A\vec{v} = \vec{0}$.

a

$a = b = c = d = 0$

Equivalent conditions for regularity

If A is regular, then

$$A\vec{v} = \vec{0} \quad \text{implies} \quad A^{-1}A\vec{v} = I_2\vec{v} = \vec{0}$$

$A^{-1}(A\vec{v}) = A^{-1}\vec{0} = \vec{0}$

$= (A^{-1}A)\vec{v} = I_2\vec{v} = \vec{v}$

Thus it follows from Theorem A that if $|A| = 0$ then A is not regular.

There exists $\exists\, \vec{v} \neq \vec{0}$, $A\vec{v} = \vec{0}$

This never happens if A is regular.

Theorem B

The following (i), (ii) and (iii) are equivalent for a 2×2 matrix A.

- **(i)** A is regular.
- **(ii)** $A\vec{v} = \vec{0} \quad \Rightarrow \quad \vec{v} = \vec{0}$
- **(iii)** $|A| \neq 0$.

(i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (i)

Proof for Theorem B

If $|A| = 0 \Rightarrow A$ is not regular.

V

- $(i) \Rightarrow (iii)$ The contraposition Not (iii) $\Rightarrow$ Not (i) is already shown.
- $(i) \Rightarrow (ii)$ Already shown.
- $(iii) \Rightarrow (i)$ Already shown.
- The contraposition Not (iii) $\Rightarrow$ Not (ii) is given in Theorem A.

(ii) $\Rightarrow$ (iii) $|A| = 0 \Rightarrow$ for some $\vec{v} \neq \vec{0}$, $A\vec{v} = \vec{0}$

contraposition of ((ii) $\Rightarrow$ (iii))

Addition of two 2×2 Matrices

Definition

Given two 2×2 matrices

$$A = (\vec{a}_1\ \vec{a}_2) = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and

$$B = (\vec{b}_1\ \vec{b}_2) = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Then the 2×2 matrix$A + B$ is defined by

$$A+B = (\vec{a}_1 + \vec{b}_1\ \vec{a}_2 + \vec{b}_2) = \begin{pmatrix} \mathbf{a}_1 + \mathbf{b}_1 \\ \mathbf{a}_2 + \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

Basic Properties (1)

Basic Properties (1)

- **(i)** $(A + B) + C = A + (B + C)$ $\rightarrow$ $A + B + C$
- **(ii)** $A + O_2 = O_2 + A = A$
- **(iii)** $A + B = B + A$
- **(iv)** $\lambda(A + B) = \lambda A + \lambda B$
- **(v)** $(\lambda + \mu)A = \lambda A + \mu A$

(i) follows from $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.
(ii) follows from $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$.
(iii) follows from $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.
(v) follows from $\cancel{(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}}$. $\cancel{\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}}$
(iv) is proved as follows.

$$\begin{aligned} LHS &= \lambda(\vec{a}_1 + \vec{b}_1\ \vec{a}_2 + \vec{b}_2) = (\lambda(\vec{a}_1 + \vec{b}_1)\ \lambda(\vec{a}_2 + \vec{b}_2)) \\ &= (\lambda\vec{a}_1 + \lambda\vec{b}_1\ \lambda\vec{a}_2 + \lambda\vec{b}_2) = (\lambda\vec{a}_1\ \lambda\vec{a}_2) + (\lambda\vec{b}_1\ \lambda\vec{b}_2) = RHS \end{aligned}$$

Basic Properties (2)

Basic Properties (2)

- **(vi)** $A(B+C) = AB + AC$
- **(vii)** $(B+C)A = BA + CA$

(vi) (i) follows from $A(\vec{b}+\vec{c}) = A\vec{b} + A\vec{c}$. In fact

$$\begin{aligned} LHS &= A(\vec{b}_1+\vec{c}_1\ \vec{b}_2+\vec{c}_2) = (A(\vec{b}_1+\vec{c}_1)\ A(\vec{b}_2+\vec{c}_2)) \\ &= (A\vec{b}_1+A\vec{c}_1\ A\vec{b}_2+A\vec{c}_2) = \underbrace{(A\vec{b}_1\ A\vec{b}_2)}_{=AB} + \underbrace{(A\vec{c}_1\ A\vec{c}_2)}_{=AC} = RHS \end{aligned}$$

(vii) (ii) follows from the identity $(B+C)\vec{a} = B\vec{a} + C\vec{a}$ which is derived as follows.

$$\begin{aligned} LHS &= (\vec{b}_1+\vec{c}_1\ \vec{b}_2+\vec{c}_2)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1(\vec{b}_1+\vec{c}_1) + a_2(\vec{b}_2+\vec{c}_2) \\ &= \cdots = \underbrace{(a_1\vec{b}_1 + a_2\vec{b}_2)}_{=B\vec{a}} + \underbrace{(a_1\vec{c}_1 + a_2\vec{c}_2)}_{=C\vec{a}} = RHS \end{aligned}$$

Theorem of Cayley and Hamilton

In the case of 2×2 matrices

Nobuyuki TOSE

October 18, 2016

Notation and a Remark

Notation $M_2(\mathbf{R})$

The set of 2×2 matrices with value in $\mathbf{R}$ is denoted by

$$M_2(\mathbf{R}) := \{A;\ 2 \times 2 \text{ matrix with value in} \mathbf{R}\}$$

Remark and Notation

We have already seen that

$$A, B \in M_2(\mathbf{R}) \ \Rightarrow AB \in M_2(\mathbf{R})$$

The nth power of $A \in M_2(\mathbf{R})$ can be defined by

$$A^n = \underbrace{A \cdots A}_{n \text{ copies}} \in M_2(\mathbf{R})$$

or recursively by

$$A^n = A^{n-1}A, \quad A^0 = I_2$$

An Application

We make it in the following two forms:

$A(A+I_2)$

$$A^2 + A = 5A + 5I_2, \quad A^2 - 5A = -A + 5I_2$$

$= A(A-5I_2)$

Accordingly we get

$$A(A+I_2) = 5(A+I_2), \quad A(A-5I_2) = -(A-5I_2)$$

We use these identities repeatedly to get

$$A^n(A+I_2) = 5^n(A+I_2), \quad A^n(A-5I_2) = (-1)^n(A-5I_2)$$

Namely

$$A^{n+1} + A^n = 5^n(A+I_2), \quad A^{n+1} - 5A^n = (-1)^n(A-5I_2)$$

This leads us to $A^n = \frac{1}{6}5^n(A+I_2) - \frac{1}{6}(-1)^n(A-5I_2)$

$$6A^n = 5^n(A+I_2) - (-1)^n(A-5I_2)$$

$$A^2(A+I_2) = A\,A(A+I_2) = A\cdot(5(A+I_2))$$
$$= 5\,A(A+I_2) = 5\cdot 5(A+I_2) = 5^2(A+I_2)$$
$$A^3(A+I_2) = A\cdot A^2(A+I_2) = A\cdot 5^2(A+I_2)$$
$$= 5^2\,A(A+I_2) = 5^2\cdot 5(A+I_2)$$

Polynomials of matrices

We can form polynomials in the matrix $A \in M_2(\mathbf{R})$.

Definition

For any polynomial

$$f(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$$

we define

$$f(A) := a_0A^n + \cdots + a_1A + a_0I_2$$

Basic properties of this operation is given in the following theorem.

Theorem

For f, g polynomials in λ, we have

$$(f+g)(A) = f(A) + g(A), \quad (f\cdot g)(A) = f(A)\cdot g(A)$$

Theorem of Cayley and Hamilton

Theorem

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$A^2 - (a+d)A + |A| \cdot I_2 = O_2$$

(proof)

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix} = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & cb+d^2 \end{pmatrix}$$

$$(a+d)A = \begin{pmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{pmatrix} = \begin{pmatrix} a^2+ad & (a+d)b \\ (a+d)c & ad+d^2 \end{pmatrix}$$

Add $|A| I_2$

$$A^2 - (a+d)A = \begin{pmatrix} bc-ad & 0 \\ 0 & bc-ad \end{pmatrix} = -|A| \cdot I_2$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A^2 - (a+d)A + (ad-bc) I_2 = O_2$

An Application

Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. Then it follows from Theorem of C-H that

$5 + (-1)$

$$A^2 - 4A - 5I_2 = O_2$$

$= (-1) \cdot 5$

We associate to A the *eigen-polynomial* of A:

$$\Phi_A(\lambda) = \lambda^2 - 4\lambda - 5 = (\lambda+1)(\lambda-5)$$

Then the Vieta's Theorem leads us to

$$5 + (-1) = 4, \quad (-1)\cdot 5 = -5$$

Thus we get

$$A^2 - ((-1)+5)A + (-1)5I_2 = O_2$$