Vectors and their operations No. 2

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October 11, 2016

Dot Product of Two Vectors

Dot Product: Definition

For two vectors $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbf{R}^n$, the dot product of $\vec{x}$ and $\vec{y}$ is defined and denoted by

$$(\vec{x}, \vec{y}) := x_1 y_1 + \cdots + x_n y_n$$

Length of Vectors: Definition

The length of $\vec{x} \in \mathbf{R}^n$ is defined and denoted by

$$||\vec{x}|| = \sqrt{(\vec{x}, \vec{x})} = \sqrt{x_1^2 + \cdots + x_n^2}$$

Basic Properties of Dot Products

(1.) (Commutativity)

$$(\vec{x}, \vec{y}) = (\vec{y}, \vec{x})$$

(2.) (Bi-linearity (i))

$$(\lambda\vec{x}, \vec{y}) = (\vec{x}, \lambda\vec{y}) = \lambda(\vec{x}, \vec{y})$$

(3.) (Bi-linearity (ii))

$$(\vec{x}+\vec{y}, \vec{z}) = (\vec{x}, \vec{z})+(\vec{y}, \vec{z}), \quad (\vec{x}, \vec{y}+\vec{z}) = (\vec{x}, \vec{y})+(\vec{x}, \vec{z})$$

(4.)

$$||\vec{x}|| \geq 0, \quad ||\vec{x}|| = 0 \Leftrightarrow \vec{x} = 0$$

An Important Formula

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For $\vec{x},\ \vec{y} \in \mathbf{R}^n$, we have

$$||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + 2(\vec{x},\vec{y}) + ||\vec{y}||^2$$

(proof)

$$\begin{aligned} LHS &= (\vec{x}+\vec{y},\vec{x}+\vec{y}) \\ &= (\vec{x},\vec{x}+\vec{y}) + (\vec{y},\vec{x}+\vec{y}) \\ &= (\vec{x},\vec{x}) + (\vec{x},\vec{y}) + (\vec{y},\vec{x}) + (\vec{y},\vec{y}) \\ &= ||\vec{x}||^2 + 2(\vec{x},\vec{y}) + ||\vec{y}||^2 = RHS \end{aligned}$$

Corollary: Pythagoras Theorem

If $\vec{x},\ \vec{y} \in \mathbf{R}^n$ are orthogonal, *i.e.* $(\vec{x},\vec{y}) = 0$, then

$$||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$$

Cauchy's Inequality

Cauchy's Inequality

$$(\vec{x}, \vec{y})^2 \leq ||\vec{x}||^2 \cdot ||\vec{y}||^2 \qquad (\#)$$

A small remark before the proof

$$||\lambda\vec{x}|| = |\lambda| \cdot ||\vec{x}||$$

(proof) (i) In case $\vec{y} = \vec{0}$, the inequality $(\#)$ is OK.
(ii) In case $\vec{y} \neq \vec{0}$, remark that $||\vec{y}||^2 > 0$. We develop $||\vec{x} - \lambda\vec{y}||^2$ as follows.

$$\begin{aligned} 0 \leq ||\vec{x} - \lambda\vec{y}||^2 &= ||\vec{x}||^2 - 2(\vec{x}, \lambda\vec{y}) + ||\lambda\vec{y}||^2 \\ &= ||\vec{x}||^2 - 2\lambda(\vec{x}, \vec{y}) + \lambda^2||\vec{y}||^2 \end{aligned}$$

Since the above inequality holds for any $\lambda \in \mathbf{R}$, the Discriminants of quadratic equation is

$$(\vec{x}, \vec{y})^2 - ||\vec{x}||^2 \cdot ||\vec{y}||^2 \leq 0$$

Triangle Inequality

Corollary to Cauchy's Inequality

$$||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$$

(proof) We have

$$||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + 2(\vec{x},\vec{y}) + ||\vec{y}||^2$$

Moreover

$$2(\vec{x},\vec{y}) \leq 2|(\vec{x},\vec{y})| \leq 2||\vec{x}|| \cdot ||\vec{y}||$$

Accordingly

$$\begin{aligned} ||\vec{x}+\vec{y}||^2 &\leq ||\vec{x}||^2 + 2||\vec{x}|| \cdot ||\vec{y}|| + ||\vec{y}||^2 \\ &= (||\vec{x}|| + ||\vec{y}||)^2 \end{aligned}$$

Application

problem

Let $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$. Then we try to find the minimum value of

$$f(t) := ||\vec{b} - t\vec{a}||^2$$

We develop $||\vec{b} - t\vec{a}||^2$ by

$$||\vec{b} - t\vec{a}||^2 = t^2||\vec{a}||^2 - 4t(\vec{a}, \vec{b}) + ||\vec{b}||^2$$

We have moreover $||\vec{a}||^2 = 7$, $(\vec{a}, \vec{b}) = 2$, $||\vec{b}||^2 = 7$. Thus

$$f(t) = 7t^2 - 2t + 7 = 7\left(t - \frac{2}{7}\right)^2 + \frac{45}{7}$$

Accordingly we find that $f(t)$ has the minimum value $\frac{45}{7}$ when $t = \frac{2}{7}$.

Problem in general

Problem

Let $\vec{a}, \vec{b} \in \mathbf{R}^n$ and assume that $\vec{a} \neq \vec{0}$. Problem is to find the minimum value of

$$f(t) = ||\vec{b} - t\vec{a}||^2$$

First we make the same approch by developing $||\vec{b} - t\vec{a}||^2$ by

$$\begin{aligned} ||\vec{b} - t\vec{a}||^2 &= t^2||\vec{a}||^2 - 2t(\vec{a}, \vec{b}) + ||\vec{b}||^2 \\ &= ||\vec{a}||^2 \left(t - \frac{(\vec{a}, \vec{b})}{||\vec{a}||^2} \right)^2 + ||\vec{b}||^2 - \frac{(\vec{a}, \vec{b})^2}{||\vec{a}||^2} \end{aligned}$$

Thus $f(t)$ takes its minimum value when $t = \frac{(\vec{a}, \vec{b})}{||\vec{a}||^2}$.

Geomtric Interpretation

In case $n = 2$ or $n = 3$, it is clear that $||\vec{b} - t\vec{a}||^2$ is minimum when

$$(\vec{b} - t\vec{a}) \perp \vec{a}$$

This condition is equivalent to

$$(\vec{b} - t\vec{a}, \vec{a}) = (\vec{b}, \vec{a}) - t||\vec{a}||^2 = 0 \quad \text{namely} \quad t = \frac{(\vec{a}, \vec{b})}{||\vec{a}||^2}$$

This is identical to the result obtained by minimizing the square functions of t.

Orthogonal Projection

Orthogonal Projection

$$\vec{w} = \frac{(\vec{a}, \vec{b})}{||\vec{a}||^2}\vec{a}$$

is called the **orthogonal projection** of $\vec{b}$ in the direction of $\vec{a}$.
$\vec{w}$ satisfies the conditions

- **(i)** $(\vec{b} - \vec{w}) \perp \vec{a}$
- **(ii)** $\vec{w} = t\vec{a}$ for some $t \in \mathbf{R}$.