I $\|\vec{a}\|^2 = 1+1+9 = 11$, $(\vec{a}, \vec{b}) = 1\cdot 2 + (-1)\cdot 1 + 3\cdot(-1)$
$= -2$

$\|\vec{b}\|^2 = 4+1+1 = 6$

Thus

$$f(t) = \|\vec{b} - t\vec{a}\|^2$$
$$= \|\vec{a}\|^2 t^2 - 2(\vec{a}, \vec{b})t + \|\vec{b}\|^2$$
$$= 11t^2 + 4t + 6$$
$$= 11\left(t + \frac{2}{11}\right)^2 + 6 - \frac{4}{11}$$
$$= 11\left(t + \frac{2}{11}\right)^2 + \frac{62}{11}$$

$f(t)$ has the minimum value $\frac{62}{11}$ when $t = -\frac{2}{11}$

II $\|\vec{a}\|^2 = 4$, $(\vec{a}, \vec{b}) = 1\cdot 2 + 1\cdot(-1) + 1\cdot 1 + (-1)\cdot 1$
$= 1$

$\|\vec{b}\|^2 = 4+1+1+1 = 7$

Thus

$$f(t) = \|\vec{a}\|^2 t^2 - 2(\vec{a}, \vec{b})t + \|\vec{b}\|^2$$
$$= 4t^2 - 2t + 7$$
$$= 4\left(t - \frac{1}{4}\right)^2 + 7 - \frac{1}{4}$$
$$= 4\left(t - \frac{1}{4}\right)^2 + \frac{27}{4}$$

Accordingly $f(t)$ has the minimum value $\frac{27}{4}$ when $t = \frac{1}{4}$.

III

$\|\vec{a}+\vec{b}+\vec{c}\|^2$

$= \|\vec{a}\|^2 + 2(\vec{a}, \vec{b}+\vec{c}) + \|\vec{b}+\vec{c}\|^2$

$= \|\vec{a}\|^2 + 2(\vec{a}, \vec{b}) + 2(\vec{a}, \vec{c})$

$+ \|\vec{b}\|^2 + 2(\vec{b}, \vec{c}) + \|\vec{c}\|^2$

$= \|\vec{a}\|^2 + \|\vec{b}\|^2 + \|\vec{c}\|^2$

$+ 2(\vec{a}, \vec{b}) + 2(\vec{b}, \vec{c}) + 2(\vec{c}, \vec{a})$

IV (1) $\|x\vec{f}_1 + y\vec{f}_2\|^2 = \|x\vec{f}_1\|^2 + 2(x\vec{f}_1, y\vec{f}_2) + \|y\vec{f}_2\|^2$

$= x^2\|\vec{f}_1\|^2 + 2xy(\vec{f}_1, \vec{f}_2) + y^2\|\vec{f}_2\|^2$

$= x^2 + y^2$

$\|x\vec{f}_1 + y\vec{f}_2 + z\vec{f}_3\|^2$

$= \|x\vec{f}_1\|^2 + \|y\vec{f}_2\|^2 + \|z\vec{f}_3\|^2$

$+ 2(x\vec{f}_1, y\vec{f}_2) + 2(y\vec{f}_2, z\vec{f}_3) + 2(z\vec{f}_3, x\vec{f}_1)$

$= x^2\|\vec{f}_1\|^2 + y^2\|\vec{f}_2\|^2 + z^2\|\vec{f}_3\|^2$

$+ 2xy(\vec{f}_1, \vec{f}_2) + 2yz(\vec{f}_2, \vec{f}_3) + 2zx(\vec{f}_3, \vec{f}_1)$

$= x^2\cdot 1 + y^2\cdot 1 + z^2\cdot 1$

$+ 2xy\cdot 0 + 2yz\cdot 0 + 2zx\cdot 0$

$= x^2 + y^2 + z^2$

(2)

$$\|\vec{g} - x\vec{f}_1 - y\vec{f}_2\|^2$$

$$= \|\vec{g}\|^2 - 2(\vec{g}, x\vec{f}_1 + y\vec{f}_2) + \|x\vec{f}_1 + y\vec{f}_2\|^2$$

$$= \|\vec{g}\|^2 - 2x(\vec{g}, \vec{f}_1) - 2y(\vec{g}, \vec{f}_2) + x^2 + y^2$$

$$= \|\vec{g}\|^2 + x^2 + y^2 - 2x(\vec{g}, \vec{f}_1) - 2y(\vec{g}, \vec{f}_2)$$

Supplementary Ex.

Let $F(x, y) = \|\vec{g} - x\vec{f}_1 - y\vec{f}_2\|$

Find the minimum value of F.

V.

$$(\vec{a}, \lambda\vec{b} + \mu\vec{c}) = \lambda(\vec{a}, \vec{b}) + \mu(\vec{a}, \vec{c})$$

$$= \lambda \cdot 0 + \mu \cdot 0 = 0.$$

Accordingly $\vec{a} \perp (\lambda\vec{b} + \mu\vec{c})$

VI. We assume that

$$c_1\vec{f}_1 + c_2\vec{f}_2 + c_3\vec{f}_3 = \vec{0}.$$

Take the dot product of the both side with $\vec{f}_3$.

Then

$$0 = (c_1\vec{f}_1 + c_2\vec{f}_2 + c_3\vec{f}_3, \vec{f}_3)$$

$$= c_1(\vec{f}_1, \vec{f}_3) + c_2(\vec{f}_2, \vec{f}_3) + c_3(\vec{f}_3, \vec{f}_3)$$

$$= c_3.$$

We have now

$$c_1 \vec{f}_1 + c_2 \vec{f}_2 = \vec{0}.$$

Take the dot product of the both side with $\vec{f}_2$.

Then

$$0 = (c_1 \vec{f}_1 + c_2 \vec{f}_2 , \vec{f}_2) = c_1 (\vec{f}_1 , \vec{f}_2) + c_2 \|\vec{f}_2\|^2$$

$$= c_1 \cdot 0 + c_2 \cdot 1 = c_2 .$$

Finaly we get

$$c_1 \vec{f}_1 = \vec{0}.$$

Since $\|\vec{f}_1\| = 1 \neq 0$, $\vec{f}_1 \neq \vec{0}$. Thus $c_1 = 0$.

We have proved so far $c_1 = c_2 = c_3 = 0$.

Accordingly $\vec{f}_1, \vec{f}_2, \vec{f}_3$ are linearly independent.

VII (1) $\vec{w} = \dfrac{(\vec{a}, \vec{e})}{\|\vec{a}\|^2} \vec{a} = \dfrac{2}{3} \vec{a}$

(2) $\vec{w} = \dfrac{(\vec{a}, \vec{e})}{\|\vec{a}\|^2} = \dfrac{3}{4} \vec{a}$

(3) $\vec{w} = \dfrac{(\vec{a}, \vec{e})}{\|\vec{a}\|^2} = \dfrac{2}{4} \vec{a} = \dfrac{1}{2} \vec{a}$

VIII

(1) $$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda+\mu \\ 0 & 1 \end{pmatrix}$$

(2) $$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda+\mu & 1 \end{pmatrix}$$

(3) $$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda\mu \end{pmatrix}$$

(4) $$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda\mu & 0 \\ 0 & 1 \end{pmatrix}$$

(5) $$\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}\begin{pmatrix} \alpha' & \gamma' \\ 0 & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & \alpha\gamma'+\gamma\beta' \\ 0 & \beta\beta' \end{pmatrix}$$

(6) $$\begin{pmatrix} \alpha & 0 \\ \gamma & \beta \end{pmatrix}\begin{pmatrix} \alpha' & 0 \\ \gamma' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & 0 \\ \gamma\alpha'+\gamma'\beta & \beta\beta' \end{pmatrix}$$

IX

(1) $$A^2 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3a \\ 0 & 1 \end{pmatrix}$$

(2) $$A^2 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2a & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 0 \\ 2a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3a & 1 \end{pmatrix}$$

(6

×

(1) $|A| = 1 \cdot 3 - 4 \cdot 2 = -5$

$$A^{-1} = -\frac{1}{5}\begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}$$

(2) $|A| = 5 \cdot 2 - 2 \cdot 2 = 6$

$$A^{-1} = \frac{1}{6}\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$$

(3) $|A| = 2 \cdot 2 - 1 \cdot 1 = 3$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

(4) $|A| = 3 \cdot 1 - 2 \cdot 1 = 1$

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$