Matrices and their operations No. 1

Multiplication of 2×2 Matrices

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Review 1: 2×2 Matrices

2×2 matrices

$$A = (\vec{a}_1 \ \vec{a}_2) = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

A 2×2 matrix is given in the following ways.

- (i) Combining two column vectors $\vec{a}_1, \ \vec{a}_2 \in \mathbf{R}^2$
- (ii) Combining two row vectors $\mathbf{a}_1$ and $\mathbf{a}_2$
- (iii) Giving 2×2 components.

NB a_{ij} is used for the component of the ith row and of the jth column.

Review 2

Multiplication of 2-dim. vectors to 2×2 matrices

$$
\begin{aligned}
A \begin{pmatrix} x \\ y \end{pmatrix} &= x\vec{a}_1 + y\vec{a}_2 \\
&= x \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} xa_{11} + ya_{12} \\ xa_{21} + ya_{22} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{a}_1 \begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{a}_2 \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}
\end{aligned}
$$

Here we use the multiplication of a row vector and a column vector defined by

$$
(\alpha \ \ \beta) \begin{pmatrix} x \\ y \end{pmatrix} = \alpha x + \beta y
$$

Review 3: Multiplication of two 2×2 matrices

Take another 2×2 matrix

$$B = (\vec{b}_1\ \vec{b}_2) = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}.$$

Then

$$AB = (A\vec{b}_1\ A\vec{b}_2) = \begin{pmatrix} \mathbf{a}_1\vec{b}_1 & \mathbf{a}_1\vec{b}_2 \\ \mathbf{a}_2\vec{b}_1 & \mathbf{a}_2\vec{b}_2 \end{pmatrix}$$

Linear Map defined by using A

A Map defined by A

Given a 2×2 matrix

$$A = (\vec{a}_1 \ \vec{a}_2) = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

we can define a map

$$F_A : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$$

$$\begin{pmatrix} s \\ t \end{pmatrix} \mapsto A \begin{pmatrix} s \\ t \end{pmatrix} = s\vec{a}_1 + t\vec{a}_2$$

Linearity of F_A

Linearity of F_A

F_A satisfies the following basic properties called **Linearity**.

- **(i)** $F_A(\vec{x}+\vec{y}) = F_A(\vec{x}) + F_A(\vec{y})$
- **(ii)** $F_A(\lambda\vec{x}) = \lambda F_A(\vec{x})$
- **(iii)** $F_A(\lambda\vec{x}+\mu\vec{y}) = \lambda F_A(\vec{x}) + \mu F_A(\vec{y})$

These three propeties are identical to the following.

- **(i)** $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
- **(ii)** $A(\lambda\vec{x}) = \lambda(A\vec{x})$
- **(iii)** $A(\lambda\vec{x}+\mu\vec{y}) = \lambda(A\vec{x}) + \mu(A\vec{y})$

Moreover remark that **(iii)** can be easily derived from **(i)** and **(ii)**. In fact

$$\begin{aligned} A(\lambda\vec{x}+\mu\vec{y}) &= A(\lambda\vec{x}) + A(\mu\vec{y}) \\ &= \lambda(A\vec{x}) + \mu(A\vec{y}) \end{aligned}$$

Proof

Proof for (i)

$$
\begin{aligned}
LHS &= A\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = A\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \\
&= (x_1 + y_1)\vec{a}_1 + (x_2 + y_2)\vec{a}_2 \\
&= x_1\vec{a}_1 + y_1\vec{a}_1 + x_2\vec{a}_2 + y_2\vec{a}_2 \\
&= (x_1\vec{a}_1 + x_2\vec{a}_2) + (y_1\vec{a}_1 + y_2\vec{a}_2) \\
&= A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = RHS
\end{aligned}
$$

Proof for (ii)

$$
\begin{aligned}
LHS &= A\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} \\
&= (\lambda x_1)\vec{a}_1 + (\lambda x_2)\vec{a}_2 \\
&= \lambda(x_1\vec{a}_1) + \lambda(x_2\vec{a}_2) \\
&= \lambda(x_1\vec{a}_1 + x_2\vec{a}_2) = RHS
\end{aligned}
$$

Associativity

Thanks to the Linearity, we can prove the following theorem about **Associativity**.

Theorem: Associativity

Given 2×2 matrices A and B. Then we have for $\vec{x} \in \mathbf{R}^2$.

$$(AB)\vec{x} = A(B\vec{x})$$

In fact,

$$\begin{aligned} RHS &= A(x_1\vec{b}_1 + x_2\vec{b}_2) \\ &= x_1(A\vec{b}_1) + x_2(A\vec{b}_2) \\ &= (A\vec{b}_1 \ A\vec{b}_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (AB)\vec{x} = LHS \end{aligned}$$

Associativity 2

Theorem: Associativity

Given 2×2 matrices A, B and C. Then

$$(AB)C = A(BC)$$

Scalar Multiplication to Matirces

Scalar Multiplication to 2×2 Matrices

Given a 2×2 matix

$$A = (\vec{a}_1\ \vec{a}_2) = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

we define a scalar multiplication by λ to A as follows:

$$\lambda A = (\lambda\vec{a}_1\ \lambda\vec{a}_2) = \begin{pmatrix} \lambda\mathbf{a}_1 \\ \lambda\mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix}$$

Scalar Multiplication to Matirces

Theorem

- **(i)** $(\lambda A)\vec{x} = \lambda(A\vec{x}) = A(\lambda\vec{x})$
- **(ii)** $(\lambda A)B = \lambda(AB) = A(\lambda B)$

The proof for **(i)** is given as follows:

$$
\begin{aligned}
(\lambda A)\vec{x} &= (\lambda\vec{a}_1 \ \lambda\vec{a}_2)\vec{x} \\
&= (\lambda x_1)\vec{a}_1 + (\lambda x_2)\vec{a}_2 \\
&= \lambda(x_1\vec{a}_1) + \lambda(x_2\vec{a}_2) \\
&= \lambda(x_1\vec{a}_1 + x_2\vec{a}_2) = \lambda(A\vec{x})
\end{aligned}
$$

Moreover the property **(ii)** is derived easily from **(i)**.

Other Basic Properties of Scalar Multiplication

Theorem

- **(iii)** $(\lambda + \mu)A = \lambda A + \mu A$
- **(iv)** $(\lambda\mu)A = \lambda(\mu A)$
- **(v)** $1A = A$ and $0A = O_2$

These properties can derived from the following corresponding properties for vectors. It is necessary to define the addition of matrices to understand (iii), and we put it off for a couple of weeks.

- **(iii)** $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$
- **(iv)** $(\lambda\mu)\vec{a} = \lambda(\mu\vec{a})$
- **(v)** $1\vec{a} = \vec{a}$ and $0\vec{a} = \vec{0}$

Special Matrices

O_2 Zero Matrix

$$O_2 = (\vec{0}\ \vec{0}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is called the zero matrix. It satisfies the identity

$$O_2X = XO_2 = O_2$$

$O_2X = O_2$ follows from

$$O_2\vec{x} = (\vec{0}\ \vec{0}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1\vec{0} + x_2\vec{0} = \vec{0}$$

and $XO_2 = O_2$ from

$$X\vec{0} = (\vec{x}_1\ \vec{x}_2) \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\vec{x}_1 + 0\vec{x}_2 = \vec{0}$$

Special Matrices 2

I_2 Identity Matrix

$$I_2 = (\vec{e}_1 \ \vec{e}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is called the **Identity Matrix**. It enjoys for a 2×2 matrix X, the identity

$$XI_2 = I_2X = X$$

It follows from the identities

$$(\vec{a} \ \vec{b})\vec{e}_1 = 1\vec{a} + 0\vec{b} = \vec{a}, \quad (\vec{a} \ \vec{b})\vec{e}_2 = 0\vec{a} + 1\vec{b} = \vec{b}$$

that

$$XI_2 = (\vec{x}_1 \ \vec{x}_2)(\vec{e}_1 \ \vec{e}_2) = (\vec{x}_1 \ \vec{x}_2) = X$$

Moreover $I_2X = X$ from

$$(\vec{e}_1 \ \vec{e}_2)\begin{pmatrix} x \\ y \end{pmatrix} = x\vec{e}_1 + y\vec{e}_2 = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Inverse matrix

We have the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

Cofactor Matrix

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its cofactor matrix is defeined by

$$\tilde{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Then we have the identity

$$A\tilde{A} = \tilde{A}A = |A| \cdot I_2$$

Inverse Matrix 2

Assume that $|A| \neq 0$. Then multiply by $\frac{1}{|A|}$ and get

$$A \cdot \frac{1}{|A|}\tilde{A} = \frac{1}{|A|}\tilde{A} \cdot A = I_2$$

Inverse Matrix

In case $|A| \neq 0$, the **Inverse Matrix** of A is defined by

$$A^{-1} = \frac{1}{|A|}\tilde{A} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Regularity of Matrices, Uniqueness of Inverse

Regularity of Matrices

A 2×2 matrix A is called **regular** if there exists another 2×2 matrix X satisfying

$$AX = XA = I_2$$

In this situation X is called the **inverse** of A.

- **(i)** If $|A| \neq 0$, A is regular.
- **(ii) (Uniqueness of the inverse)** Assume

$$AX = XA = I_2, \quad AY = YA = I_2$$

Then $X = Y$. In fact, from $AX = I_2$ multiplied by Y from the left follows

$$Y(AX) = YI_2 = Y$$

On the other hand, $Y(AX) = (YA)X = I_2X = X$.
Accordingly $X = Y$.

In case $|A| = 0$

Theorem A

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a 2×2 matrix with $|A| = 0$. Then there exists $\vec{v} \neq \vec{0}$ satisfying $A\vec{v} = \vec{0}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d \\ -c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- **(i)** In case $d \neq 0$ or $c \neq 0$, $\begin{pmatrix} d \\ -c \end{pmatrix} \neq \vec{0}$ AND $A \begin{pmatrix} d \\ -c \end{pmatrix} = \vec{0}$.
- **(ii)** In case $b \neq 0$ or $c \neq 0$, $\begin{pmatrix} -b \\ a \end{pmatrix} \neq \vec{0}$ AND $A \begin{pmatrix} -b \\ a \end{pmatrix} = \vec{0}$.
- **(iii)** Not (i) AND Not (ii). Then $A = O_2$.

Equivalent conditions for regularity

If A is regular, then

$$A\vec{v} = \vec{0} \quad \text{implies} \quad \vec{v} = \vec{0}.$$

In fact by multiplying A^{-1} to $A\vec{v} = \vec{0}$ to get

$$A^{-1}A\vec{v} = A^{-1}\vec{0} \text{ namely } \vec{v} = I_2\vec{v} = \vec{0}$$

Thus it follows from Theorem A that if $|A| = 0$ then A is not regular.

Theorem B

The following (i), (ii) and (iii) are equivalent for a 2×2 matrix A.

- **(i)** A is regular.
- **(ii)** $A\vec{v} = \vec{0} \quad \Rightarrow \quad \vec{v} = \vec{0}$
- **(iii)** $|A| \neq 0$.

Proof for Theorem B

- $(i) \Rightarrow (iii)$ The contraposition Not (iii) $\Rightarrow$ Not (i) is already shown.
- $(i) \Rightarrow (ii)$ Already shown.
- $(iii) \Rightarrow (i)$ Already shown.
- The contraposition Not (iii) $\Rightarrow$ Not (ii) is given in Theorem A.

Addition of two 2×2 Matrices

Definition

Given two 2×2 matrices

$$A = (\vec{a}_1 \ \vec{a}_2) = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and

$$B = (\vec{b}_1 \ \vec{b}_2) = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Then the 2×2 matrix$A + B$ is defined by

$$A + B = (\vec{a}_1 + \vec{b}_1 \ \vec{a}_2 + \vec{b}_2) = \begin{pmatrix} \mathbf{a}_1 + \mathbf{b}_1 \\ \mathbf{a}_2 + \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

Basic Properties (1)

Basic Properties (1)

- **(i)** $(A+B)+C = A+(B+C)$
- **(ii)** $A + O_2 = O_2 + A = A$
- **(iii)** $A + B = B + A$
- **(iv)** $\lambda(A+B) = \lambda A + \lambda B$
- **(v)** $(\lambda+\mu)A = \lambda A + \mu A$

(i) follows from $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$.
(ii) follows from $\vec{a}+\vec{0} = \vec{0}+\vec{a} = \vec{a}$.
(iii) follows from $\vec{a}+\vec{b} = \vec{b}+\vec{a}$.
(v) follows from $(\lambda+\mu)\vec{a} = \lambda\vec{a}+\mu\vec{a}$.
(iv) is proved as follows.

$$\begin{aligned} LHS &= \lambda(\vec{a}_1+\vec{b}_1 \ \ \vec{a}_2+\vec{b}_2) = (\lambda(\vec{a}_1+\vec{b}_1) \ \ \lambda(\vec{a}_2+\vec{b}_2)) \\ &= (\lambda\vec{a}_1+\lambda\vec{b}_1 \ \ \lambda\vec{a}_2+\lambda\vec{b}_2) = (\lambda\vec{a}_1 \ \ \lambda\vec{a}_2) + (\lambda\vec{b}_1 \ \ \lambda\vec{b}_2) = RHS \end{aligned}$$

Basic Properties (2)

Basic Properties (2)

- **(vi)** $A(B+C) = AB + AC$
- **(vii)** $(B+C)A = BA + CA$

(vi) follows from $A(\vec{b}+\vec{c}) = A\vec{b} + A\vec{c}$. In fact

$$\begin{aligned} LHS &= A(\vec{b}_1+\vec{c}_1\ \ \vec{b}_2+\vec{c}_2) = (A(\vec{b}_1+\vec{c}_1)\ \ A(\vec{b}_2+\vec{c}_2)) \\ &= (A\vec{b}_1 + A\vec{c}_1\ \ A\vec{b}_2 + A\vec{c}_2) = (A\vec{b}_1\ \ A\vec{b}_2) + (A\vec{c}_1\ \ A\vec{c}_2) = RHS \end{aligned}$$

(vii) follows from the identity $(B+C)\vec{a} = B\vec{a} + C\vec{a}$ which is derived as follows.

$$\begin{aligned} LHS &= (\vec{b}_1+\vec{c}_1\ \ \vec{b}_2+\vec{c}_2)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1(\vec{b}_1+\vec{c}_1) + a_2(\vec{b}_2+\vec{c}_2) \\ &= \cdots = (a_1\vec{b}_1 + a_2\vec{b}_2) + (a_1\vec{c}_1 + a_2\vec{c}_2) = RHS \end{aligned}$$