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Vectors and their operations No. 2

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Dot Product of Two Vectors

Dot Product: Definition

For two vectors $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbf{R}^n$, the dot product of $\vec{x}$ and $\vec{y}$ is defined and denoted by

$$(\vec{x}, \vec{y}) := x_1 y_1 + \cdots + x_n y_n$$

Length of Vectors: Definition

The length of $\vec{x} \in \mathbf{R}^n$ is defined and denoted by

$$||\vec{x}|| = \sqrt{(\vec{x}, \vec{x})} = \sqrt{x_1^2 + \cdots + x_n^2}$$

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Basis Properties of Dot Products

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(1.) (Commutativity)

$$(\vec{x}, \vec{y}) = (\vec{y}, \vec{x})$$

(2.) (Bi-linearity (i))

$$(\lambda\vec{x}, \vec{y}) = (\vec{x}, \lambda\vec{y}) = \lambda(\vec{x}, \vec{y})$$

(3.) (Bi-linearity (ii))

$$(\vec{x}+\vec{y}, \vec{z}) = (\vec{x}, \vec{z})+(\vec{y}, \vec{z}), \quad (\vec{x}, \vec{y}+\vec{z}) = (\vec{x}, \vec{y})+(\vec{x}, \vec{z})$$

(4.)

$$||\vec{x}|| \geq 0, \quad ||\vec{x}|| = 0 \Leftrightarrow \vec{x} = 0$$

An Important Formula

An Important Formula

For $\vec{x}, \ \vec{y} \in \mathbf{R}^n$, we have

$$||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + 2(\vec{x}, \vec{y}) + ||\vec{y}||^2$$

(proof)

$$\begin{aligned} LHS &= (\vec{x}+\vec{y}, \vec{x}+\vec{y}) \\ &= (\vec{x}, \vec{x}+\vec{y}) + (\vec{y}, \vec{x}+\vec{y}) \\ &= (\vec{x}, \vec{x}) + (\vec{x}, \vec{y}) + (\vec{y}, \vec{x}) + (\vec{y}, \vec{y}) \\ &= ||\vec{x}||^2 + 2(\vec{x}, \vec{y}) + ||\vec{y}||^2 = RHS \end{aligned}$$

Corollary: Pythagoras Theorem

If $\vec{x}, \ \vec{y} \in \mathbf{R}^n$ are orthogonal, *i.e.* $(\vec{x}, \vec{y}) = 0$, then

$$||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$$

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Cauchy's Inequality

Cauchy's Inequality

$$(\vec{x}, \vec{y})^2 \leq ||\vec{x}||^2 \cdot ||\vec{y}||^2 \qquad (\#)$$

A small remark before the proof

$$||\lambda\vec{x}|| = |\lambda| \cdot ||\vec{x}||$$

$(\vec{x}, \vec{0})^2 = 0$

$||\vec{x}||^2 \cdot ||\vec{0}||^2 = 0$

(proof) (i) In case $\vec{y} = \vec{0}$, the inequality $(\#)$ is OK.
(ii) In case $\vec{y} \neq \vec{0}$, remark that $||\vec{y}||^2 > 0$. We develop $||\vec{x} - \lambda\vec{y}||^2$ as follows.

$$\begin{aligned} 0 \leq ||\vec{x} - \lambda\vec{y}||^2 &= ||\vec{x}||^2 - 2(\vec{x}, \lambda\vec{y}) + ||\lambda\vec{y}||^2 \\ &= ||\vec{x}||^2 - 2\lambda(\vec{x}, \vec{y}) + \lambda^2||\vec{y}||^2 \end{aligned}$$

$||\lambda\vec{y}|| = |\lambda| \cdot ||\vec{y}||$

Since the above inequality holds for any $\lambda \in \mathbf{R}$, the Discriminants of quadratic equation is

$$(\vec{x}, \vec{y})^2 - ||\vec{x}||^2 \cdot ||\vec{y}||^2 \leq 0 \rightarrow (\vec{x}, \vec{y})^2 \leq ||\vec{x}||^2 \cdot ||\vec{y}||^2$$

$\vec{y} = \vec{0} \iff ||\vec{y}|| = 0$

Triangle Inequality

Corollary to Cauchy's Inequality

$$||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$$

(proof) We have

$\leq ||\vec{x}|| \cdot ||\vec{y}||$

$$||\vec{y} + \vec{y}||^2 = ||\vec{x}||^2 + 2(\vec{x}, \vec{y}) + ||\vec{y}||^2$$

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Moreover

$$2(\vec{x}, \vec{y}) \leq 2|(\vec{x}, \vec{y})| \leq 2||\vec{x}|| \cdot ||\vec{y}||$$

Accordingly

$$\begin{aligned} ||\vec{x} + \vec{y}||^2 &\leq ||\vec{x}||^2 + 2||\vec{x}|| \cdot ||\vec{y}|| + ||\vec{y}||^2 \\ &= (||\vec{x}|| + ||\vec{y}||)^2 \end{aligned}$$

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Application

problem

Let $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$. Then we try to find the minimum value of

$$f(t) := ||\vec{b} - t\vec{a}||^2$$

We develop $||\vec{b} - t\vec{a}||^2$ by

$$||\vec{b} - t\vec{a}||^2 = t^2||\vec{a}||^2 - 2t(\vec{a}, \vec{b}) + ||\vec{b}||^2$$

$1 \cdot (-1) + 1 \cdot 2$
$+ (-1) \cdot 1 + 2 \cdot 1$
$= 2$

We have moreover $||\vec{a}||^2 = 7$, $(\vec{a}, \vec{b}) = 2$, $||\vec{b}||^2 = 7$. Thus

$$f(t) = 7t^2 - 2\cdot 2t + 7 = 7\left(t - \frac{2}{7}\right)^2 + \frac{45}{7}$$

Accordingly we find that $f(t)$ has the minimum value $\frac{45}{7}$ when $t = \frac{2}{7}$.

Problem in general

Problem

Let $\vec{a}, \vec{b} \in \mathbf{R}^n$ and assume that $\vec{a} \neq \vec{0}$. Problem is to find the minimum value of

$$f(t) = ||\vec{b} - t\vec{a}||^2$$

First we make the same approch by developing $||\vec{b} - t\vec{a}||^2$ by

$$||\vec{b} - t\vec{a}||^2 = t^2||\vec{a}||^2 - 2t(\vec{a}, \vec{b}) + ||\vec{b}||^2$$
$$= ||\vec{a}||^2\left(t - \frac{(\vec{a}, \vec{b})}{||\vec{a}||^2}\right)^2 + ||\vec{b}||^2 - \frac{(\vec{a}, \vec{b})^2}{||\vec{a}||^2}$$

minimum value.

Thus $f(t)$ takes its minimum value when $t = \frac{(\vec{a}, \vec{b})}{||\vec{a}||^2}$.

(14)

Geomtric Interpretation

In case $n = 2$ or $n = 3$, it is clear that $||\vec{b} - t\vec{a}||^2$ is minimum when

$$(\vec{b} - t\vec{a}) \perp \vec{a}$$

This condition is equivallent to

$$(\vec{b} - t\vec{a}, \vec{a}) = (\vec{b}, \vec{a}) - t||\vec{a}||^2 = 0 \quad \text{namely} \quad t = \frac{(\vec{a}, \vec{b})}{||\vec{a}||^2}$$

This is identical to the result obtained by minimizing the square functions of t.

Orthogonal Projection

Orthogonal Projection

$$\vec{w} = \frac{(\vec{a}, \vec{b})}{||\vec{a}||^2}\vec{a}$$

is called the **orthogonal projection** of $\vec{b}$ in the direction of $\vec{a}$. $\vec{w}$ satisfies the conditions

- **(i)** $(\vec{b} - \vec{w}) \perp \vec{a}$
- **(ii)** $\vec{w} = t\vec{a}$ for some $t \in \mathbf{R}$.

$||\vec{c} - s\vec{a} - t\vec{e}||^2$ $\quad s, t \in \mathbb{R}$.

Matrices and their operations No. 1
Multiplication of 2×2 Matrices

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Review 1: 2×2 Matrices

2×2 matrices

$$A = (\vec{a}_1 \ \vec{a}_2) = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

A 2×2 matrix is given in the following ways.

- (i) Combining two column vectors $\vec{a}_1, \ \vec{a}_2 \in \mathbf{R}^2$
- (ii) Combining two row vectors $\mathbf{a}_1$ and $\mathbf{a}_2$
- (iii) Giving 2×2 components.

NB a_{ij} is used for the component of the ith row and of the jth column.

$= (\vec{a}_1\ \vec{a}_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Review 2

Multiplication of 2-dim. vectors to 2 × 2 matrices

$(a_{11}\ a_{12})\begin{pmatrix} x \\ y \end{pmatrix}$

$$
\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= x\vec{a}_1 + y\vec{a}_2 \\
&= x\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} xa_{11} + ya_{12} \\ xa_{21} + ya_{22} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{a}_1\begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{a}_2\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}
\end{aligned}
$$

$= (a_{21}\ a_{22})\begin{pmatrix} x \\ y \end{pmatrix}$

Here we use the multiplication of a row vector and a column vector defined by

$$(\alpha\ \beta)\begin{pmatrix} x \\ y \end{pmatrix} = \alpha x + \beta y$$

Review 3: Multiplication of two 2 × 2 matrices

Take another 2 × 2 matrix

$$B = (\vec{b}_1\ \vec{b}_2) = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}.$$

Then

$$AB = (A\vec{b}_1\ A\vec{b}_2) = \begin{pmatrix} \mathbf{a}_1\vec{b}_1 & \mathbf{a}_1\vec{b}_2 \\ \mathbf{a}_2\vec{b}_1 & \mathbf{a}_2\vec{b}_2 \end{pmatrix}$$

Linear Map defined by using A

A Map defined by A

Given a 2×2 matrix

$$A = (\vec{a}_1 \ \vec{a}_2) = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

we can define a map

$$F_A : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$$
$$\begin{pmatrix} s \\ t \end{pmatrix} \mapsto A \begin{pmatrix} s \\ t \end{pmatrix} = s\vec{a}_1 + t\vec{a}_2$$

Linearity of F_A

Linearity of F_A

F_A satisfies the following basic properties called **Linearity**.

- **(i)** $F_A(\vec{x} + \vec{y}) = F_A(\vec{x}) + F_A(\vec{y})$
- **(ii)** $F_A(\lambda\vec{x}) = \lambda F_A(\vec{x})$
- **(iii)** $F_A(\lambda\vec{x} + \mu\vec{y}) = \lambda F_A(\vec{x}) + \mu F_A(\vec{y})$

These three propeties are identical to the following.

- **(i)** $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
- **(ii)** $A(\lambda\vec{x}) = \lambda(A\vec{x})$
- **(iii)** $A(\lambda\vec{x} + \mu\vec{y}) = \lambda(A\vec{x}) + \mu(A\vec{y})$

Moreover remark that **(iii)** can be easily derived from **(i)** and **(ii)**. In fact

(i)

$$\begin{aligned} A(\lambda\vec{x} + \mu\vec{y}) &= A(\lambda\vec{x}) + A(\mu\vec{y}) \\ &= \lambda(A\vec{x}) + \mu(A\vec{y}) \end{aligned}$$

(ii)

Proof

Proof for (i)

$(\vec{a}_1\ \vec{a}_2)$

For $\vec{a} \in \mathbb{R}^n$

$(\lambda+\mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$

$$LHS = A\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = A\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}$$
$$= (x_1+y_1)\vec{a}_1 + (x_2+y_2)\vec{a}_2$$
$$= x_1\vec{a}_1 + y_1\vec{a}_1 + x_2\vec{a}_2 + y_2\vec{a}_2$$
$$= (x_1\vec{a}_1 + x_2\vec{a}_2) + (y_1\vec{a}_1 + y_2\vec{a}_2)$$
$$= A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = RHS$$

Proof for (ii)

$(\vec{a}_1\ \vec{a}_2)$

$(\lambda\mu)\vec{a} = \lambda(\mu\vec{a})$

$$LHS = A\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$$
$$= (\lambda x_1)\vec{a}_1 + (\lambda x_2)\vec{a}_2$$
$$= \lambda(x_1\vec{a}_1) + \lambda(x_2\vec{a}_2)$$
$$= \lambda(x_1\vec{a}_1 + x_2\vec{a}_2) = RHS$$

$\lambda\left(A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$

$\lambda(\vec{a}+\vec{b}) = \lambda\vec{a} + \lambda\vec{b}$

Associativity

Thanks to the Linearity, we can prove the following theorem about **Associativity**.

Theorem: Associativity

Given 2×2 matrices A and B. Then we have for $\vec{x} \in \mathbf{R}^2$.

$$(AB)\vec{x} = A(B\vec{x})$$

$AB\vec{x}$

In fact,

$$RHS = A(x_1\vec{b}_1 + x_2\vec{b}_2)$$
(iii) $$= x_1(A\vec{b}_1) + x_2(A\vec{b}_2)$$
$$= (A\vec{b}_1\ A\vec{b}_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (AB)\vec{x} = RHS$$

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Associativity 2

Theorem: Associativity

Given 2×2 matrices A, B and C. Then

$$(AB)C = A(BC)$$

$$\text{RHS} = \left((AB)\vec{c}_1 \quad (AB)\vec{c}_2\right)$$

$$= \left(A(B\vec{c}_1) \quad A(B\vec{c}_2)\right)$$

By the last theorem

$$= A\left(B\vec{c}_1 \quad B\vec{c}_2\right)$$

$$= A(BC) = \text{RHS}.$$

Scalar Multiplication to Matirces

Scalar Multiplication to 2×2 Matrices

Given a 2×2 matix

$$A = (\vec{a}_1 \ \vec{a}_2) = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

we define a scalar multiplication by λ to A as follows:

$$\lambda A = (\lambda\vec{a}_1 \ \lambda\vec{a}_2) = \begin{pmatrix} \lambda\mathbf{a}_1 \\ \lambda\mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix}$$