Vectors and their operations

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Review 1

Determinants for pairs of 2dim. vectors

For $\vec{a}, \vec{b} \in \mathbf{R}^2$

$$|\vec{a}\ \vec{b}| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

2×2 matrices

$$A = (\vec{a}_1\ \vec{a}_2) = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

A 2×2 matrix is given in the following ways.

- (i) Combining two column vectors $\vec{a}_1,\ \vec{a}_2 \in \mathbf{R}^2$
- (ii) Combining two row vectors $\mathbf{a}_1$ and $\mathbf{a}_2$
- (iii) Giving 2×2 components.

NB a_{ij} is used for the component of the ith row and of the jth column.

Review 2

Multiplication of 2-dim. vectors to 2×2 matrices

$$
\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= x\vec{a} + y\vec{a}_2 \\
&= x\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} xa_{11} + ya_{12} \\ xa_{21} + ya_{22} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{a}_1 \begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{a}_2 \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}
\end{aligned}
$$

Here we use the multiplication of a row vector and a column vector defined by

$$
(\alpha\ \ \beta)\begin{pmatrix} x \\ y \end{pmatrix} = \alpha x + \beta y
$$

Review 3: Multiplication of two 2×2 matrices

Take another 2×2 matrix

$$B = (\vec{b}_1\ \vec{b}_2) = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}.$$

Then

$$AB = (A\vec{b}_1\ A\vec{b}_2) = \begin{pmatrix} \mathbf{a}_1\vec{b}_1 & \mathbf{a}_1\vec{b}_2 \\ \mathbf{a}_2\vec{b}_1 & \mathbf{a}_2\vec{b}_2 \end{pmatrix}$$

Definition

$\mathbf{R}^n$ is the set of all n-dimensional column vectors.

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbf{R}^n$$

Take another $\vec{y} \in \mathbf{R}^n$. Then

$$\vec{x} \pm \vec{y} = \begin{pmatrix} x_1 \pm y_1 \\ x_2 \pm y_2 \\ \vdots \\ x_n \pm y_n \end{pmatrix}, \quad \lambda\vec{x} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

Basic Properties of the vector operations (1)

1a. (commutativity)

$$\vec{x} + \vec{y} = \vec{y} + \vec{x}$$

1b. (associativity)

$$(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$$

1c.

$$\vec{x} + \vec{0} = \vec{x}$$

1d.

$$\vec{x} + (-\vec{x}) = \vec{0}$$

Basic Properties of the vector operations (2)

2a.
$$1\vec{x} = \vec{x}$$

2b.
$$\lambda(\mu\vec{x}) = (\lambda\mu)\vec{x}$$

3a.
$$(\lambda + \mu)\vec{x} = \lambda\vec{x} + \mu\vec{x}$$

3b.
$$\lambda(\vec{x} + \vec{y}) = \lambda\vec{x} + \lambda\vec{y}$$

Linear independence for vectors

$$\vec{z}_1, \cdots, \vec{z}_s \in \mathbf{R}^n$$

Definition

$\vec{z}_1, \cdots, \vec{z}_s$ are **linearly independent** iff

$$\lambda_1 \vec{z}_1 + \cdots + \lambda_s \vec{z}_s = \vec{0} \quad \Longrightarrow \quad \lambda_1 = \cdots = \lambda_s = 0$$

Definition

$\vec{z}_1, \cdots, \vec{z}_s$ are **linearly dependent** iff

$$\lambda_1 \vec{z}_1 + \cdots + \lambda_s \vec{z}_s = \vec{0}$$

with some $\lambda_i \neq 0$.

In case $s = 2$, *i.e.* two vectors

Assume that $\vec{a}, \vec{b} \in \mathbf{R}^n$ are linearly dependent.

$$x\vec{a} + y\vec{b} = \vec{0} \quad \text{AND} \quad (x \neq 0 \text{ OR } y \neq 0)$$

In case $x \neq 0$,

$$\vec{a} = -\frac{y}{x}\vec{b}$$

In case $y \neq 0$,

$$\vec{b} = -\frac{x}{y}\vec{a}$$

Conclusion

- $\vec{a}$ and $\vec{b}$ are linearly dependent iff $\vec{a} \parallel \vec{b}$.
- $\vec{a}$ and $\vec{b}$ are linearly independent iff $\vec{a} \not\parallel \vec{b}$.

Linear (In)Dependence and Linear Equations

- In case $s = 2$, $n = 2$, i.e. 2 dimensional case,

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \ \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Then

$$x\vec{a} + y\vec{b} = \vec{0} \Leftrightarrow \left\{ \begin{array}{ccccc} xa_1 & + & yb_1 & = & 0 \\ xa_2 & + & yb_2 & = & 0 \end{array} \right.$$

- In case $s = 2$, $n = 3$, i.e. 3 dimensional case,

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \ \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Then

$$x\vec{a} + y\vec{b} = \vec{0} \Leftrightarrow \left\{ \begin{array}{ccccc} xa_1 & + & yb_1 & = & 0 \\ xa_2 & + & yb_2 & = & 0 \\ xa_3 & + & yb_3 & = & 0 \end{array} \right.$$

Cramer's Rule

If $D := \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \neq 0$,

$$\begin{cases} xa_1 + yb_1 = \alpha \\ xa_2 + yb_2 = \beta \end{cases} \Rightarrow x = \frac{1}{D}\begin{vmatrix} \alpha & b_1 \\ \beta & b_2 \end{vmatrix},\ y = \frac{1}{D}\begin{vmatrix} a_1 & \alpha \\ a_2 & \beta \end{vmatrix}$$

We consider the **homogeneous equations** *i.e.* $\alpha = \beta = 0$.

Homogeneous equations

If $D \neq 0$,

$$\begin{cases} xa_1 + yb_1 = 0 \\ xa_2 + yb_2 = 0 \end{cases} \Rightarrow x = y = 0$$

Theorems on linear (in)dependence and determinants

We have proved the following Theorem 1.

Theorem 1

If $D := \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \neq 0$, then

$$\vec{a},\ \vec{b} \text{ are linearly independent.}$$

The following Theorem 2 shows that the inverse of Theorem 1 holds. Theorem 2 is particularly important in higher leveled mathematics for economists, such as eigen-value problems.

Theorem 2 (proof is to be given in a couple of weeks)

If $D := \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = 0$, then

$$\vec{a},\ \vec{b} \text{ are linearly dependent.}$$

Example

Let $\vec{a} = \begin{pmatrix} t-2 \\ -4 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -3 \\ t-1 \end{pmatrix}$, then

$$\begin{aligned} |\vec{a}\ \vec{b}| &= \begin{vmatrix} t-2 & -3 \\ -4 & t-1 \end{vmatrix} \\ &= (t-2)(t-1)-(-4)(-3) = t^2-3t-10 = (t-5)(t+2). \end{aligned}$$

Accordingly

$$\vec{a}, \vec{b} \text{ are linearly independent} \Leftrightarrow t \neq -2 \text{ AND } t \neq 5 \qquad (1)$$

$$\vec{a}, \vec{b} \text{ are linearly dependent} \Leftrightarrow t = -2 \text{ OR } t = 5 \qquad (2)$$

What happens when $t = -2$ or $t = 5$?

In case of general n-dimensional vectors, we have the following Theorem 3. Let

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in \mathbf{R}^n$$

Theorem 3

If $\begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \neq 0$ for some i, j with $i \neq j$, then

$$\vec{a},\ \vec{b} \text{ are linearly independent.}$$

Example

Consider

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix} \in \mathbf{R}^3$$

The it follows from

$$\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} \neq 0$$

that

$$\vec{a},\ \vec{b} \quad \text{are linearly independent.}$$

In case of $s = 3$, three vectors.

Let $\vec{a},\ \vec{b},\ \vec{c} \in \mathbf{R}^n$.

Linearly independent case

$$\begin{aligned} &\vec{a}, \vec{b}, \vec{c} \text{ are linearly independent} \\ \Leftrightarrow\ &\left(x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \quad \Rightarrow x = y = z = 0\right) \end{aligned}$$

Linearly dependent case

$$\begin{aligned} &\vec{a}, \vec{b}, \vec{c} \text{ are linearly dependent} \\ \Leftrightarrow\ &\left(x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \text{ AND } ((x \neq 0) \text{ OR } (y \neq 0) \text{ OR } (z \neq 0))\right) \end{aligned}$$

In case $x \neq 0$, we get

$$\vec{a} = -\frac{y}{x}\vec{b} - \frac{z}{x}\vec{c}$$

Example

Let $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}.$

Then

$$\begin{aligned} & x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \\ \Leftrightarrow & \left\{ \begin{array}{rcrcrcll} x & - & 2y & + & 7z & = & 0 & \cdots (i) \\ 2x & + & 3y & + & 4z & = & 0 & \cdots (ii) \\ x & - & 2y & + & 7z & = & 0 & \cdots (iii) \end{array} \right. \\ \Leftrightarrow & \left\{ \begin{array}{rcrcrcll} x & - & 2y & + & 7z & = & 0 & \cdots (i) \\ & & 7y & - & 10z & = & 0 & \cdots (ii)' = (ii) - (i) \times (-2) \end{array} \right. \end{aligned}$$

$(y, z) = (10, 7)$ satisfies $(ii)'$. Moreover it follows from (i) that $x = 2y - 7z = 2 \times 10 - 7 \times 7 = -29$. Thus we have

$$-29\vec{a} + 10\vec{b} + 7\vec{c} = \vec{0}$$

Example

Let $\vec{a} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}0\\-1\\3\end{pmatrix}, \quad \vec{c} = \begin{pmatrix}2\\-1\\2\end{pmatrix}.$

Then

$$
\begin{aligned}
& x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \\
\Leftrightarrow\ & \left\{\begin{array}{rcrcrcll}
x & & & + & 2z & = & 0 & \cdots (i) \\
2x & - & y & - & z & = & 0 & \cdots (ii) \\
3x & + & 3y & + & 2z & = & 0 & \cdots (iii)
\end{array}\right. \\
\Leftrightarrow\ & \left\{\begin{array}{rcrcrcll}
x & & & + & 2z & = & 0 & \cdots (i) \\
 & - & y & - & 5z & = & 0 & \cdots (ii)' = (ii) - (i) \times (-2) \\
 & & 3y & - & 4z & = & 0 & \cdots (iii)' = (iii) - (i) \times (-3)
\end{array}\right.
\end{aligned}
$$

It follows from $\begin{vmatrix}-1 & -5\\ 3 & -4\end{vmatrix} = 19 \neq 0$ that $(ii)'$ AND $(iii)'$ implies $y = z = 0$. Moreover by (i) $x = -2z = -2 \times 0 = 0$. Thus $(i), (ii), (iii)$ implies $x = y = z = 0$.