VIII

(i) It follows from $\begin{vmatrix} 3 & 1 \\ 4 & -3 \end{vmatrix} = 3\cdot(-3) - 4\cdot 1 = -13 \neq 0$ that $\vec{u}, \vec{v}$ are linearly independent.

(ii) $3\vec{u} = 3\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \end{pmatrix} = \vec{v}$. Thus we have $3\vec{u} - \vec{v} = \vec{0}$.
$\vec{u}, \vec{v}$ are linearly dependent.

(iii) It follows from $\begin{vmatrix} 4 & 2 \\ 3 & -6 \end{vmatrix} = 4(-6) - 3\cdot 2 = -30 \neq 0$ that $\vec{u}, \vec{v}$ are linearly independent.

(iv) $-\frac{1}{2}\vec{u} = -\frac{1}{2}\begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \vec{v}$ Thus we have
$-\frac{1}{2}\vec{u} - \vec{v} = \vec{0}$. $\vec{u}, \vec{v}$ are linearly dependent.

IX (i)

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \iff \begin{cases} x + 2y + 7z = 0 & (1) \\ -2x + y - 4z = 0 & (2) \\ x - y + z = 0 & (3) \end{cases}$$

$$\iff \begin{cases} x + 2y + 7z = 0 & (1) \\ 5y + 10z = 0 & (2)' = (2) + (1)\times 2 \\ -3y - 6z = 0 & (3)' = (3) - (1) \end{cases}$$

It follows $\begin{vmatrix} 5 & 10 \\ -3 & -6 \end{vmatrix} = 5(-6) - (-3)\cdot 10 = 0$ that there exsts $(x, y) \neq (0, 0)$ satisfying $(2)'$ and $(3)'$. In fact $(x, y) = (2, -1)$ satisfies $(2)'$ and $(3)'$. Then

$x = -2y - 7z = -2\cdot 2 - 7\cdot(-1) = 3$ by (i). Accordingly we have

$$3\vec{a} + 2\vec{b} - \vec{c} = \vec{0}$$

This shows that $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent.

(ii)

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \iff \begin{cases} x + y + 2z = 0 & (1) \\ 2x - 3y - z = 0 & (2) \\ -3x + 2y + 5z = 0 & (3) \end{cases}$$

$$\iff \begin{cases} x + y + 2z = 0 & (1) \\ -5y - 5z = 0 & (2)' = (2) + (1) \times (-2) \\ 5y + 11z = 0 & (3)' = (3) + (1) \times 3 \end{cases}$$

It follows from $\begin{vmatrix} -5 & -5 \\ 5 & 11 \end{vmatrix} = -5 \cdot 11 - 5 \cdot (-5) = -30 \neq 0$ that $(2)'$ and $(3)'$ imply $y = z = 0$. Then

$$x = -y - 2z = -0 - 2 \cdot 0 = 0$$

by (1). Accordingly $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ implies $x = y = z = 0$, i.e. $\vec{a}, \vec{b}, \vec{c}$ are linearly independent.

(iii) $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$

$$\iff \begin{cases} x + 2y + 3z + 2w = 0 & (1) \\ -3x - z + 4w = 0 & (2) \\ 7x - 6y - z - 5w = 0 & (3) \end{cases}$$

$$\iff \begin{cases} x + 2y + 3z + 2w = 0 & (1) \\ 6y + 8z + 10w = 0 & (2)' = (2) + (1) \times 3 \\ 8y - 22z - 19w = 0 & (3)' = (3) + (1) \times (-7) \end{cases}$$

$$\iff \begin{cases} x + 2y + 3z + 2w = 0 & (1) \\ 24y + 32z + 40w = 0 & (2)'' = (2)' \times 4 \\ 24y - 66z - 57w = 0 & (3)'' = (3)' \times 3 \end{cases}$$

$$\iff \begin{cases} x + 2y + 3z + 2w = 0 & (1) \\ 24y + 32z + 40w = 0 & (2)'' \\ -98z - 97w = 0 & (3)''' = (3)'' - (2)'' \end{cases}$$

$(z, w) = (97, -98)$ statisfies $(3)'''$. Then

$$y = -\frac{1}{24}(32z + 40w)$$

$$= -\frac{4}{3}z - \frac{5}{3}w$$

$$= -\frac{4}{3}\cdot 97 + \frac{5}{3}\cdot 98 = 34,$$

Finally we get

$$x = -2y - 3z - 2w$$

$$= -2\cdot 34 - 3\cdot 97 - 2(-98)$$

$$= -163.$$

Thus $(x, y, z, w) = (-163, 34, 97, -98)$ satisfies $(1), (2), (3)$ and we have the relation

$$-163\vec{a} + 34\vec{b} + 97\vec{c} - 98\vec{d} = \vec{o}.$$

Accordingly $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are linearly dependent.

(iv) $0\cdot\vec{a} + 1\cdot\vec{b} + 0\cdot\vec{c} = \vec{o} + 1\cdot\vec{o} + \vec{o} = \vec{o}$

Thus $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent.

Assume that we have a linear relation (4)

$$c_1(\vec{u}+\vec{v}) + c_2(\vec{u}-\vec{v}) + c_3(\vec{u}-2\vec{v}+\vec{w}) = \vec{0}$$

This is equivallent to

$$(c_1+c_2+c_3)\,\vec{u} + (c_1-c_2-2c_3)\,\vec{v} + c_3\,\vec{w} = \vec{0}.$$

Since $\vec{u}, \vec{v}, \vec{w}$ are linearly indepent, it follows that

$$c_1+c_2+c_3=0 \quad \cdots(1), \qquad c_1-c_2-2c_3=0 \quad \cdots(2), \qquad c_3=0. \quad \cdots(3)$$

$$\Longleftrightarrow \begin{cases} c_1+c_2=0 & \cdots(1)' \\ c_1-c_2=0 & \cdots(2)' \\ c_3=0 & \cdots(3) \end{cases}$$

It follows from $(1)'+(2)'$ that

$$2c_1=0$$

and from $(1)'-(2)'$ that

$$2c_2=0.$$

Accordingly we have $c_1=c_2=c_3=0$.

XI

(i) $x\vec{u} + y\vec{v} = 0 \iff \begin{cases} x + 4y = 0 & (1) \\ 2x + 3y = 0 & (2) \\ 3x + 2y = 0 & (3) \\ 4x + y = 0 & (4) \end{cases}$

Then it follows for $\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3 - 8 = -5 \neq 0$ that (1) an (2) imply $x = y = 0$. Accordingly $\vec{u}$ and $\vec{v}$ are linealy independent.

(ii) $-\frac{1}{2}\vec{u} = -\frac{1}{2}\begin{pmatrix} -1 \\ 6 \\ -12 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -3 \\ 6 \end{pmatrix} = \vec{v}$

Thus we have

$$\frac{1}{2}\vec{u} + \vec{v} = \vec{0}$$

This shows that $\vec{u}, \vec{v}$ are linealy dependent.

XII Assume

$$c_1 \alpha \vec{a} + c_2 \beta \vec{b} + c_3 \gamma \vec{c} = \vec{0}$$

Since $\vec{a}, \vec{b}, \vec{c}$ are linealy independent, it follows that

$$c_1\alpha = c_2\beta = c_3\gamma = 0.$$

Moreover since $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$, we have

$$c_1 = c_2 = c_3 = 0$$

Thus $\alpha\vec{a}, \beta\vec{b}, \gamma\vec{c}$ are linealy independent.