Vectors and their operations

Nobuyuki TOSE

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Review 1

Determinants for pairs of 2dim. vectors

For $\vec{a}, \vec{b} \in \mathbf{R}^2$

$$|\vec{a}\ \vec{b}| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

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2×2 matrices

$$A = (\vec{a}_1\ \vec{a}_2) = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

1st column.

2nd row

A 2×2 matrix is given in the following ways.

- (i) Combining two column vectors $\vec{a}_1,\ \vec{a}_2 \in \mathbf{R}^2$
- (ii) Combining two row vectors $\mathbf{a}_1$ and $\mathbf{a}_2$
- (iii) Giving 2×2 components.

NB a_{ij} is used for the component of the ith row and of the jth column.

Review 2

Multiplication of 2-dim. vectors to 2 × 2 matrices

$$
\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= x\vec{a} + y\vec{b} \\
&= x\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} xa_{11} + ya_{12} \\ \boxed{xa_{21} + ya_{22}} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{a}_1 \begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{a}_2 \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}
\end{aligned}
$$

$\vec{a_2}$

$A = (\vec{a_1}\ \vec{a_2}) = \cancel{\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}}$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$= (a_{21}\ a_{22})\begin{pmatrix} x \\ y \end{pmatrix}$

$= \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix}$

Here we use the multiplication of a row vector and a column vector defined by

$$(\alpha\ \beta)\begin{pmatrix} x \\ y \end{pmatrix} = \alpha x + \beta y$$

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \left(1 \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + 0 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + 1 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right)$$

$$= \begin{pmatrix} a_1 & \lambda a_1 + b_1 \\ a_2 & \lambda a_2 + b_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 1 \cdot a_1 + \lambda a_2 & 1 \cdot b_1 + \lambda \cdot b_2 \\ 0 \cdot a_1 + 1 \cdot a_2 & 0 \cdot b_1 + 1 \cdot b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + \lambda a_2 & b_1 + \lambda b_2 \\ a_2 & b_2 \end{pmatrix}$$

Review 3: Multiplication of two 2×2 matrices

Take another 2×2 matrix

$$B = (\vec{b}_1\ \vec{b}_2) = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}.$$

Then

$$AB = (A\vec{b}_1\ A\vec{b}_2) = \begin{pmatrix} \mathbf{a}_1\vec{b}_1 & \mathbf{a}_1\vec{b}_2 \\ \mathbf{a}_2\vec{b}_1 & \mathbf{a}_2\vec{b}_2 \end{pmatrix}$$

Definition

$\mathbf{R}^n$ is the set of all n-dimensional column vectors.

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbf{R}^n$$

Take another $\vec{y} \in \mathbf{R}^n$. Then

$$\vec{x} \pm \vec{y} = \begin{pmatrix} x_1 \pm y_1 \\ x_2 \pm y_2 \\ \vdots \\ x_n \pm y_n \end{pmatrix}, \quad \lambda\vec{x} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

Basic Properties of the vector operations (1)

1a. (commutativity)

$$\vec{x} + \vec{y} = \vec{y} + \vec{x}$$

1b. (associativity)

$$(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z}) = \vec{x} + \vec{y} + \vec{z}$$

1c.

$$\vec{x} + \vec{0} = \vec{x}$$

$$\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

1d.

$$\vec{x} + (-\vec{x}) = \vec{0}$$

$$-\vec{x} = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix}$$

Basic Properties of the vector operations (2)

2a.

$$1\vec{x} = \vec{x}$$

2b.

$$\lambda(\mu\vec{x}) = (\lambda\mu)\vec{x}$$

3a.

$$(\lambda + \mu)\vec{x} = \lambda\vec{x} + \mu\vec{x}$$

3b.

$$\lambda(\vec{x} + \vec{y}) = \lambda\vec{x} + \mu\vec{y}$$

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Linear independence for vectors

$s = 2, 3$

$$\vec{z}_1, \cdots, \vec{z}_s \in \mathbf{R}^n$$

Definition

$\vec{z}_1, \cdots, \vec{z}_s$ are **linearly independent** iff

$$\lambda_1\vec{z}_1 + \cdots + \lambda_s\vec{z}_s = \vec{0} \quad \Longrightarrow \quad \lambda_1 = \cdots = \lambda_s = 0$$

Definition

$\vec{z}_1, \cdots, \vec{z}_s$ are **linearly dependent** iff

$$\lambda_1\vec{z}_1 + \cdots + \lambda_s\vec{z}_s = \vec{0}$$

with some $\lambda_i \neq 0$.

In case $s = 2$, *i.e.* two vectors

Assume that $\vec{a}, \vec{b} \in \mathbf{R}^n$ are linearly dependent.

$$x\vec{a} + y\vec{b} = \vec{0} \quad \text{AND} \quad (x \neq 0 \text{ OR } y \neq 0)$$

In case $x \neq 0$,

$$\vec{a} = -\frac{y}{x}\vec{b}$$

In case $y \neq 0$,

x

$$\vec{b} = -\frac{y}{x}\vec{a}$$

y

$//$

Conclusion

- $\vec{a}$ and $\vec{b}$ are linearly dependent iff $\vec{a} \parallel \vec{b}$.
- $\vec{a}$ and $\vec{b}$ are linearly independent iff $\vec{a} \nparallel \vec{b}$.

Linear (In)Dependence and Linear Equations

- In case $s = 2$, $n = 2$, i.e. 2 dimensional case,

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

  Then

$$x\vec{a} + y\vec{b} = \vec{0} \Leftrightarrow \left\{ \begin{array}{ccccc} xa_1 & + & yb_1 & = & 0 \\ xa_2 & + & yb_2 & = & 0 \end{array} \right.$$

- In case $s = 2$, $n = 3$, i.e. 3 dimensional case,

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

  Then

$$x\vec{a} + y\vec{b} = \vec{0} \Leftrightarrow \left\{ \begin{array}{ccccc} xa_1 & + & yb_1 & = & 0 \\ xa_2 & + & yb_2 & = & 0 \\ xa_3 & + & yb_3 & = & 0 \end{array} \right.$$

Cramer's Rule

$$\begin{vmatrix} 0 & b_1 \\ 0 & b_2 \end{vmatrix} = 0 \cdot b_2 - 0 \cdot b_1 = 0$$

$$\begin{vmatrix} a_1 & 0 \\ a_2 & 0 \end{vmatrix} = 0$$

Cramer's Rule

If $D := \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \neq 0$,

$$\begin{cases} xa_1 + yb_1 = \alpha \\ xa_2 + yb_2 = \beta \end{cases} \Rightarrow x = \frac{1}{D}\begin{vmatrix} \alpha & b_1 \\ \beta & b_2 \end{vmatrix}, \; y = \frac{1}{D}\begin{vmatrix} a_1 & \alpha \\ a_2 & \beta \end{vmatrix}$$

We consider the **homogeneous equations** *i.e.* $\alpha = \beta = 0$.

Homogeneous equations

If $D \neq 0$,

$$\boxed{\begin{cases} xa_1 + yb_1 = 0 \\ xa_2 + yb_2 = 0 \end{cases}} \Rightarrow x = y = 0$$

$$\Updownarrow$$

$$x\vec{a} + y\vec{b} = \vec{0}$$

Theorems on linear (in)dependence and determinants

We have proved the following Theorem 1. $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

Theorem 1

If $D := \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \neq 0$, then

$\vec{a}, \vec{b}$ are linearly independent.

The following Theorem 2 shows that the inverse of Theorem 1 holds. Theorem 2 is particularly important in higher leveled mathematics for economists, such as eigen-value problems.

Theorem 2 (proof is to be given in a couple of weeks)

If $D := \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = 0$, then

$\vec{a}, \vec{b}$ are linearly dependent.

$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2 \neq 0.$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ are linearly indep.

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \parallel \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 2 = 0.$

Example

Let $\vec{a} = \begin{pmatrix} t-2 \\ -4 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -3 \\ t-1 \end{pmatrix}$, then

$$
\begin{aligned}
|\vec{a}\ \vec{b}| &= \begin{vmatrix} t-2 & -3 \\ -4 & t-1 \end{vmatrix} \\
&= (t-2)(t-1) - (-4)(-3) = t^2 - 3t - 10 = (t-5)(t+2).
\end{aligned}
$$

Accordingly

$$\vec{a}, \vec{b} \text{ are linearly independent} \Leftrightarrow t \neq -2 \text{ AND } t \neq 5 \qquad (1)$$

$$\vec{a}, \vec{b} \text{ are linearly dependent} \Leftrightarrow t = -2 \text{ OR } t = 5 \qquad (2)$$

What happens when $t = -2$ or $t = 5$?

In case $t = -2$ $\begin{pmatrix} -4 & -3 \\ -4 & -3 \end{pmatrix}$

$t = 5$ $\begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix}$

General Theorems on lin. (in)dependence and det.

In case of general n-dimensional vectors, we have the following Theorem 3. Let

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in \mathbf{R}^n$$

Theorem 3

If $\begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \neq 0$ for some i, j with $i \neq j$, then

$$\vec{a},\ \vec{b} \text{ are linearly independent.}$$

$$x\vec{a} + y\vec{b} = \vec{0} \qquad = \begin{pmatrix} \vdots \\ x a_i + y b_i \\ \vdots \\ x a_j + y b_j \\ \vdots \end{pmatrix} \begin{array}{l} \leftarrow i\text{th component} \\ \\ \leftarrow j\text{th component.} \end{array}$$

$$\longrightarrow \begin{cases} x a_i + y b_i = 0 \\ x a_j + y b_j = 0 \end{cases} \xrightarrow[\text{Th.1}]{} \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \neq 0. \quad x = y = 0$$

Example

Consider

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix} \in \mathbf{R}^3$$

The it follows from

$$\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} \neq 0$$

that

$$\vec{a},\ \vec{b} \quad \text{are linearly independent.}$$

In case of $s = 3$, three vectors.

Let $\vec{a},\ \vec{b},\ \vec{c} \in \mathbf{R}^n$.

Linearly independent case

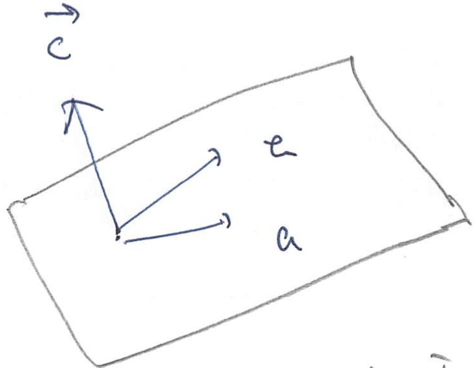
$$\vec{a}, \vec{b}, \vec{c} \text{ are linearly independent}$$
$$\Leftrightarrow \left(x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \quad \Rightarrow x = y = z = 0 \right)$$

Linearly dependent case

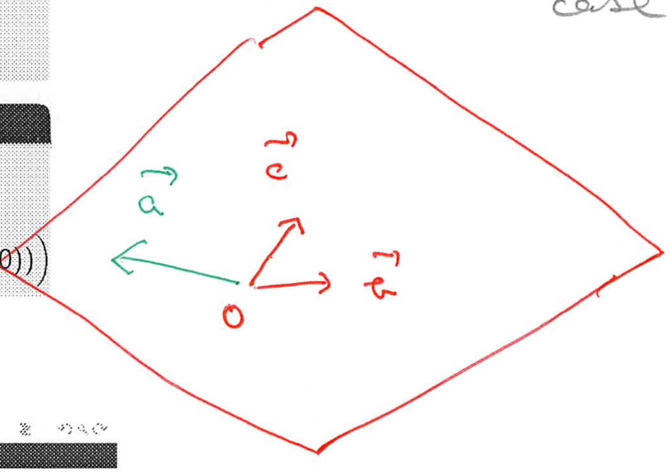
$$\vec{a}, \vec{b}, \vec{c} \text{ are linearly dependent}$$
$$\Leftrightarrow \left(x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \text{ AND } ((x \neq 0) \text{ OR } (y \neq 0) \text{ OR } (z \neq 0)) \right)$$

In case $x \neq 0$, we get

$$\vec{a} = -\frac{y}{x}\vec{b} - \frac{z}{x}\vec{c}$$



Linearly independent case.



Linearly dependent case

Example

Let $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$.

Then

$$
\begin{aligned}
& x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \\
\Leftrightarrow\ & \begin{cases} x - 2y + 7z = 0 & \cdots (i) \\ 2x + 3y + 4z = 0 & \cdots (ii) \\ x - 2y + 7z = 0 & \cdots (iii) \end{cases} \\
\Leftrightarrow\ & \begin{cases} x - 2y + 7z = 0 & \cdots (i) \\ 7y - 10z = 0 & \cdots (ii)' = (ii) - (i) \times (-2) \end{cases}
\end{aligned}
$$

$(y, z) = (10, 7)$ satisfies $(ii)'$. Moreover it follows from (i) that $x = 2y - 7z = 2 \times 10 - 7 \times 7 = -29$. Thus we have

$$-29\vec{a} + 10\vec{b} + 7\vec{c} = \vec{0}$$

(x, y, z)
$= (-29, 10, 7)$.

Example

Let $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$

Then

$$
\begin{aligned}
& x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \\
\Leftrightarrow & \left\{ \begin{array}{rcrcrcl}
x & & & + & 2z & = & 0 \quad \cdots (i) \\
2x & - & y & - & z & = & 0 \quad \cdots (ii) \\
3x & + & 3y & + & 2z & = & 0 \quad \cdots (iii)
\end{array} \right. \\
\Leftrightarrow & \left\{ \begin{array}{rcrcrcl}
x & & & + & 2z & = & 0 \quad \cdots (i) \\
 & - & y & - & 5z & = & 0 \quad \cdots (ii)' = (ii) - (i) \times (-2) \\
 & & 3y & - & 4z & = & 0 \quad \cdots (iii)' = (iii) - (i) \times (-3)
\end{array} \right.
\end{aligned}
$$

It follows from $\begin{vmatrix} -1 & -5 \\ 3 & -4 \end{vmatrix} = 19 \neq 0$ that $(ii)'$ AND $(iii)'$ implies $y = z = 0$. Moreover by (i) $x = -2z = -2 \times 0 = 0$. Thus $(i), (ii), (iii)$ ~~is~~ implies $x = y = z = 0$.