I

$$\begin{cases} ax + by = \alpha & (1) \\ cx + dy = \beta & (2) \end{cases}$$

We consider $(1) \times c - (2) \times a$

$$\begin{array}{rll} & acx + bcy = \alpha c & (1) \times c \\ -) & acx + ady = \beta a & (2) \times a \\ \hline & (bc - ad)\, y = \alpha c - \beta a. & \end{array}$$

Thus we get

$$(ad - bc)\, y = a\beta - c\alpha$$

This is written by determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} y = \begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix}$$

II (1) $\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3 \cdot 5 - (-2) \cdot 4$

$= 15 + 8 = 23$

(2) $\begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} = 2 \cdot 1 - 4 \cdot 5 = 2 - 20 = -18$

(3) $\begin{vmatrix} 6 & 1 \\ 3 & -2 \end{vmatrix} = 6 \cdot (-2) - 3 \cdot 1 = -15$

III (i)

$$\begin{vmatrix} t-2 & -3 \\ -4 & t-1 \end{vmatrix} = (t-2)(t-1) - (-3)(-4)$$

$$= t^2 - 3t + 2 - 12$$

$$= t^2 - 3t - 10$$

$$= (t-5)(t+2)$$

(ii)

$$\begin{vmatrix} t-5 & 7 \\ -1 & t+3 \end{vmatrix} = (t-5)(t+3) - (-1)\cdot 7$$

$$= t^2 - 2t - 15 + 7$$

$$= t^2 - 2t - 8$$

$$= (t-4)(t+2)$$

IV (i) $D = \begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix} = 3\cdot(-2) - 4\cdot 5$

$= -6-20 = -26 \neq 0$

We can apply Cramer's rule.

$x = -\frac{1}{26} \begin{vmatrix} 8 & 5 \\ 1 & -2 \end{vmatrix} = -\frac{1}{26}\,(8\cdot(-2) - 1\cdot 5)$

$= -\frac{1}{26}(-21) = \frac{21}{26}$

$y = -\frac{1}{26} \begin{vmatrix} 3 & 8 \\ 4 & 1 \end{vmatrix} = -\frac{1}{26}\,(3\cdot 1 - 4\cdot 8)$

$= -\frac{1}{26}(-35) = \frac{35}{26}$

(ii) $D = \begin{vmatrix} 2 & -3 \\ 4 & 7 \end{vmatrix} = 2\cdot 7 - 4\,(-3) = 14+12$

$= 26 \neq 0$

We can apply Cramer's rule.

$x = \frac{1}{26} \begin{vmatrix} -1 & -3 \\ -1 & 7 \end{vmatrix} = \frac{1}{26}\,((-1)\cdot 7 - (-1)(-3))$

$= \frac{1}{26}(-10) = -\frac{5}{13}$

$y = \frac{1}{26} \begin{vmatrix} 2 & -1 \\ 4 & -1 \end{vmatrix} = \frac{1}{26}\,(2\cdot(-1) - 4\cdot(-1))$

$= \frac{1}{26}\cdot 2 = \frac{1}{13}$

V The system of equations is equivallent to

$$\begin{cases} x + 2y = -z - 2 \\ 2x - y = z + 1 \end{cases}$$

$$D = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5 \neq 0$$

We can apply Cramer's rule as follows.

$$x = \frac{1}{-5} \begin{vmatrix} -z-2 & 2 \\ z+1 & -1 \end{vmatrix}$$

$$= -\frac{1}{5} \left( (-z-2)(-1) - (z+1) \cdot 2 \right)$$

$$= -\frac{1}{5} \left( z + 2 - 2z - 2 \right) = -\frac{1}{5}(-z)$$

$$= \frac{z}{5}$$

$$y = \frac{1}{-5} \begin{vmatrix} 1 & -z-2 \\ 2 & z+1 \end{vmatrix} = -\frac{1}{5} \left( z+1 - 2(-z-2) \right)$$

$$= -\frac{1}{5} (3z + 5)$$

VI (iii)

$$LHS = \begin{vmatrix} a_1 & b_1 + c_1 \\ a_2 & b_2 + c_2 \end{vmatrix}$$

$$= a_1 (b_2 + c_2) - a_2 (b_1 + c_1)$$

$$= a_1 b_2 - a_2 b_1 + a_1 c_2 - a_2 c_1$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= |\vec{a}\ \vec{b}| + |\vec{a}\ \vec{c}| \quad = RHS$$

(iv)

$$LHS = \begin{vmatrix} a_1 & \lambda_1 b_1 \\ a_2 & \lambda_2 b_2 \end{vmatrix} = a_1 (\lambda b_2) - a_2 (\lambda b_1)$$

$$= \lambda ( a_1 b_2 - a_2 b_1 ) = \lambda \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = RHS$$

VII

(i) $$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \beta b_1 \\ \alpha a_2 & \beta b_2 \end{pmatrix}$$

(ii) $$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \alpha b_1 \\ \beta a_2 & \beta b_2 \end{pmatrix}$$

(iii) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

(iv) $$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

(v) $$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\theta' - \sin\theta\sin\theta' & -\cos\theta\sin\theta' - \sin\theta\cos\theta' \\ \sin\theta\cos\theta' + \cos\theta\sin\theta' & -\sin\theta\sin\theta' + \cos\theta\cos\theta' \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\theta') & -\sin(\theta+\theta') \\ \sin(\theta+\theta') & \cos(\theta+\theta') \end{pmatrix}$$

(vi) $$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$