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$$\begin{cases} ax + by = \alpha & \cdots (1) \\ cx + dy = \beta & \cdots (2) \end{cases}$$

$a, b, c, d \cdots$

$\alpha, \beta, \gamma, \delta, \varepsilon, \cdots$

$(1) \times d - (2) \times b$

$$\begin{array}{rl} & adx + bdy = \alpha d \quad \cdots (1) \times d \\ -) & bcx + bdy = \beta b \quad \cdots (2) \times b. \\ \hline & (ad - bc)x = \alpha d - \beta b \end{array}$$

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Determinants of the pair $\vec{a}$ and $\vec{b}$

$$|\vec{a} \ \vec{b}| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$-$ $+$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} x = \begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} y = \begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix} \leftarrow \text{Ex. 1.}$$

We assume

$$D := \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

$$x = \frac{1}{D} \begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix}, \quad y = \frac{1}{D} \begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix}$$

Cramer's Rule

Example

$$\begin{cases} 2x - 3y = 7 \\ 3x + 5y = 1 \end{cases}$$

$$D = \begin{vmatrix} 2 & -3 \\ 3 & 5 \end{vmatrix} = 2 \cdot 5 - 3 \cdot (-3) = 19. \neq 0.$$

$$x = \frac{1}{19} \begin{vmatrix} 7 & -3 \\ 1 & 5 \end{vmatrix} = \frac{1}{19} \left(7 \times 5 - 1 \cdot (-3) \right) = \frac{1}{19} \times 38 = 2.$$

$$y = \frac{1}{19} \begin{vmatrix} 2 & 7 \\ 3 & 1 \end{vmatrix} = \frac{1}{19} (2 \times 1 - 3 \times 7) = \frac{1}{19} (-19) = -1.$$

Application (x, y, z)

$$\begin{cases} x + y - z = 1 & \cdots (1) \\ 2x - y + 2z = -1 & \cdots (2) \end{cases}$$

Problem Express x and y by z.

$$\begin{cases} x + y = z + 1 \\ 2x - y = -2z - 1 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 1 \times (-1) - 2 \times 1 = -3 \neq 0.$$

$$x = \frac{1}{-3} \begin{vmatrix} z+1 & 1 \\ -2z-1 & -1 \end{vmatrix} = -\frac{1}{3} \Big((z+1)(-1) - (-2z-1) \cdot 1 \Big)$$

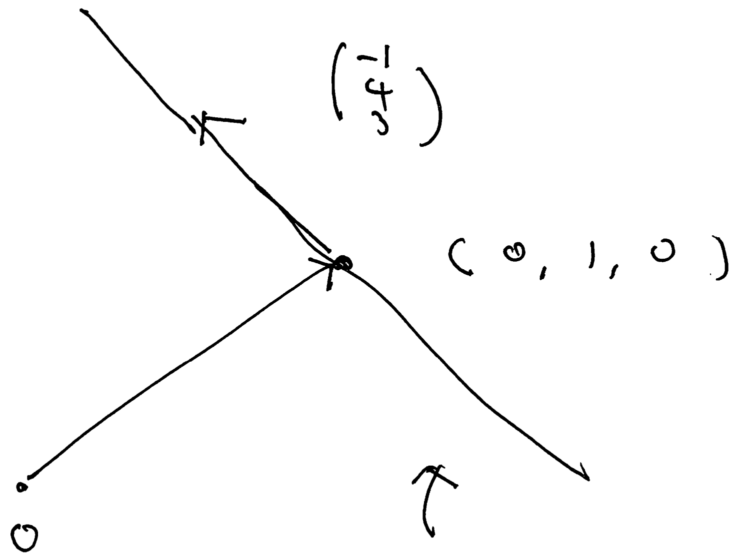
$$= -\frac{1}{3} z.$$

$$y = \frac{1}{-3} \begin{vmatrix} 1 & z+1 \\ 2 & -2z-1 \end{vmatrix} = \frac{1}{3} (4z+3)$$

calculate by yourself.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}z \\ \frac{1}{3}(4z+3) \\ z \end{pmatrix} = \frac{z}{3}\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$z = 0.$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$



$\{ (x, y, z) ;\ (1) \text{ and } (2) \}$

(i) (1) and (2) express a line.

(ii) The line passes through $(0, 1, 0)$

(iii) The Directional vector is $\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$.

Basic properties of determinants.

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \text{etc.}$$

and $\lambda \in \mathbb{R}$.

(i) $|\vec{a}+\vec{b} \;\; \vec{c}| = |\vec{a} \;\; \vec{c}| + |\vec{b} \;\; \vec{c}|$

(ii) $|\lambda \vec{a} \;\; \vec{b}| = \lambda |\vec{a} \;\; \vec{b}|$

} Linearity wrt 1st column.

(iii) $|\vec{a} \;\; \vec{b}+\vec{c}| = |\vec{a} \;\; \vec{b}| + |\vec{a} \;\; \vec{c}|$

(iv) $|\vec{a} \;\; \lambda\vec{b}| = \lambda |\vec{a} \;\; \vec{b}|$

} Linearity wrt 2nd column

Exercise VI

(i) $$|\vec{a}+\vec{b} \;\; \vec{c}| = \begin{vmatrix} a_1+b_1 & c_1 \\ a_2+b_2 & c_2 \end{vmatrix}$$

$$= (a_1+b_1)c_2 - (a_2+b_2)\times c_1$$

$$= (a_1c_2 - a_2c_1) + (b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = |\vec{a} \;\; \vec{c}| + |\vec{b} \;\; \vec{c}|$$

(ii) $$|\lambda\vec{a}\ \vec{b}| = \begin{vmatrix} \lambda a_1 & b_1 \\ \lambda a_2 & b_2 \end{vmatrix} = \lambda a_1 b_2 - \lambda a_2 b_1$$

$$= \lambda(a_1 b_2 - a_2 b_1) = \lambda \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \lambda |\vec{a}\ \vec{b}|$$

2 × 2 Matrices

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

A 2×2 matrix is formed by combining two column vectors

$$A = (\vec{a} \ \vec{b}) = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

1st row

2nd row   1st column   2nd column

$$A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (\vec{a} \ \vec{b}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \vec{a} + \beta \vec{b}$$

$$= \alpha \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \beta \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha a_1 + \beta b_1 \\ \alpha a_2 + \beta b_2 \end{pmatrix}$$

$$(\vec a\ \vec b)\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\text{" } \vec e_1}=1\cdot\vec a+0\cdot\vec b=\vec a$$

$$(\vec a\ \vec b)\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\text{" } \vec e_2}=0\cdot\vec a+1\cdot\vec b=\vec b$$

standard unit vectors

$$(\vec a\ \vec b)\begin{pmatrix}1\\\lambda\end{pmatrix}=1\cdot\vec a+\lambda\vec b=\vec a+\lambda\vec b=\begin{pmatrix} & \end{pmatrix}$$

$$A=(\vec a\ \vec b)=\begin{pmatrix}a_1 & b_1\\a_2 & b_2\end{pmatrix}$$

$$X=(\vec x\ \vec y)=\begin{pmatrix}x_1 & y_1\\x_2 & y_2\end{pmatrix}$$

$$(\vec a\ \vec b)\begin{pmatrix}x_1\\x_2\end{pmatrix}\qquad(\vec a\ \vec b)\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

$$AX=(A\vec x\ \ A\vec y)=(\underline{x_1\vec a+x_2\vec b}\quad y_1\vec a+y_2\vec b)$$

2x2 matrix

$$=x_1\begin{pmatrix}a_1\\a_2\end{pmatrix}+x_2\begin{pmatrix}b_1\\b_2\end{pmatrix}$$

$$=\begin{pmatrix}x_1a_1+x_2b_1 & y_1a_1+y_2b_1\\x_1a_2+x_2b_2 & y_1a_2+y_2b_2\end{pmatrix}$$

$$=y_1\begin{pmatrix}a_1\\a_2\end{pmatrix}+y_2\begin{pmatrix}b_1\\b_2\end{pmatrix}$$

Examples

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1 \quad 0} \\ \boxed{0 \quad 1} \end{pmatrix}\begin{pmatrix} \boxed{\begin{matrix} a_1 \\ a_2 \end{matrix}} & \boxed{\begin{matrix} b_1 \\ b_2 \end{matrix}} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

$$(a_1 \ b_1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad (a_1 \ b_1)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$AX = \begin{pmatrix} \boxed{x_1 a_1 + x_2 b_1} & \boxed{y_1 a_1 + y_2 b_1} \\ \boxed{x_1 a_2 + x_2 b_2} & \boxed{y_1 a_2 + y_2 b_2} \end{pmatrix}$$

$$(a_2 \ b_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad (a_2 \ b_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$(\alpha \ \beta)\begin{pmatrix} \gamma \\ \delta \end{pmatrix} := \alpha\gamma + \beta\delta$$

$$I_2 = E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ identity matrix.

X 2×2 matrix $\qquad I_2 X = X I_2 = X$

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b_1 & a_1 \\ b_2 & a_2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ a_1 & b_1 \end{pmatrix}$$