intro 2016

06/20

$$A = \begin{pmatrix} -1 & 4 \\ 4 & 5 \end{pmatrix}$$

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda+1 & -4 \\ -4 & \lambda-5 \end{vmatrix} = (\lambda+1)(\lambda-5)-16$$

$$\shortparallel$$

$$|\lambda I_2 - A| \qquad = \lambda^2 - 4\lambda - 21$$

$$= (\lambda-7)(\lambda+3)$$

から A の固有値は $\lambda = -3, 7$.

固有ベクトルを求める.

$\lambda = -3$ のとき. $A\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix} \iff (-3I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff \begin{pmatrix} \boxed{-2 \quad -4} \\ \boxed{-4 \quad -8} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$

$-2x-4y$

$-4x-8y.$

$\iff x + 2y = 0$

より $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $(y \neq 0)$ が固有ベクトル

$$\boxed{\vec{r_1} \text{ は } y = \frac{1}{\sqrt{5}} \qquad A\vec{r_1} = -3\vec{r_1} \text{ が成立.}}$$

$\lambda=7$ のとき. $A\begin{pmatrix}x\\y\end{pmatrix}=7\begin{pmatrix}x\\y\end{pmatrix} \iff (7I_2-A)\begin{pmatrix}x\\y\end{pmatrix}=\vec{0} \iff \begin{pmatrix}8&-4\\-4&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\vec{0}$

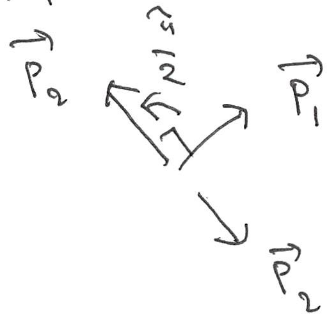
$\iff 2x-y=0$

より $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\2x\end{pmatrix}=x\begin{pmatrix}1\\2\end{pmatrix}$ $(x\neq 0)$ が固有ベクトル.

$\vec{r_2}$ は $x=-\frac{1}{\sqrt{5}}$ に相当　$A\vec{r_2}=7\vec{r_2}$

$\frac{1}{\sqrt{5}}\begin{pmatrix}-2&1\\1&2\end{pmatrix}$ は回転ではない　$\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$

回転



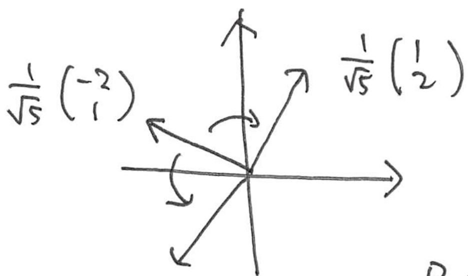
$(\vec{p_1}\ \ -\vec{p_2})$ が回転.

A 対称.　$A\vec{v_1}=\alpha\vec{v_1}$
$\alpha\neq\beta$　$A\vec{v_2}=\beta\vec{v_2}$ $\rightarrow$ $\vec{v_1}\perp\vec{v_2}$

$\vec{p_1}=\frac{1}{\|\vec{v_1}\|}\vec{v_1}$, $\vec{p_2}=\frac{1}{\|\vec{v_2}\|}\vec{v_2}$

$(\vec{p_1}\ \vec{p_2})$ or $(\vec{p_1}\ -\vec{p_2})$ が回転

$\vec{p_1}\perp\vec{p_2}$



$R=\frac{1}{\sqrt{5}}\begin{pmatrix}-2&-1\\1&-2\end{pmatrix}$ は回転行列.

$=(\vec{r_1}\ \ \vec{r_2})$

$$A R = (A\vec{r}_1 \ \ A\vec{r}_2) = (-3\vec{r}_1 \ \ 7\vec{r}_2)$$

$$= (\vec{r}_1 \ \ \vec{r}_2)\begin{pmatrix} -3 & 0 \\ 0 & 7 \end{pmatrix} = R\begin{pmatrix} -3 & 0 \\ 0 & 7 \end{pmatrix}$$

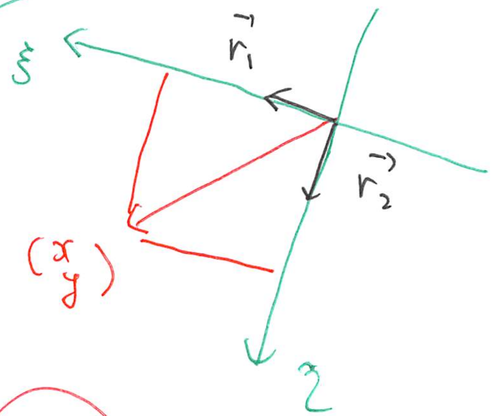
Rは回転で正則だから $R^{-1}AR = \begin{pmatrix} -3 & 0 \\ 0 & 7 \end{pmatrix}$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

逆変換

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = -x^2 + 8xy + 5y^2$$

$$A = \begin{pmatrix} -1 & 4 \\ 4 & 5 \end{pmatrix}$$



$$\begin{pmatrix} x \\ y \end{pmatrix} = \xi\vec{r}_1 + \eta\vec{r}_2 = R\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

で書き換える

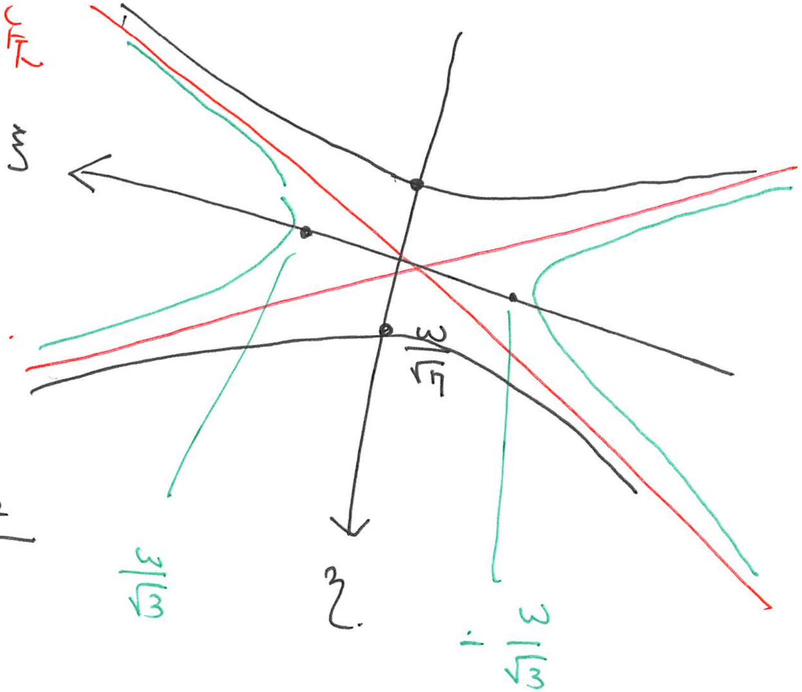
$$= \left(R^{-1}A\begin{pmatrix} x \\ y \end{pmatrix}, R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\underbrace{R^{-1}AR}_{= \begin{pmatrix} -3 & 0 \\ 0 & 7 \end{pmatrix}} \underbrace{R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}}_{= \begin{pmatrix} \xi \\ \eta \end{pmatrix}}, \underbrace{R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}}_{= \begin{pmatrix} \xi \\ \eta \end{pmatrix}}\right)$$

R^{-1} は逆回転

$$= -3\xi^2 + 7\eta^2$$

$$-3\xi^2+7\eta^2=C$$ 曲線

双曲線族



1) $\underline{C=0}$

$$\eta=\pm\sqrt{\frac{3}{7}}\,\xi.$$

2) $C=\omega^2>0 \quad (\omega>0)$

$$7\eta^2=\omega^2+3\xi^2$$

$$\eta^2=\frac{\omega^2+3\xi^2}{7}\to\eta=\pm\sqrt{\frac{\omega^2+3\xi^2}{7}}$$

3) $C=-\omega^2<0 \quad (\omega>0)$

$$7\eta^2=-\omega^2+3\xi^2$$

$$\eta^2=\frac{3\xi^2-\omega^2}{7}$$

$$\eta=\pm\sqrt{\frac{3\xi^2-\omega^2}{7}}$$

$$3\xi^2-\omega^2\geqq 0.$$

$$|\xi|\geqq\frac{\omega}{\sqrt{3}}$$

$$(A\vec{r}_1, \vec{r}_1) = (-3\vec{r}_1, \vec{r}_1) = -3\underbrace{\|\vec{r}_1\|}_{=1}{}^2 = -3 < 0$$

$$(A\vec{r}_2, \vec{r}_2) = (7\vec{r}_2, \vec{r}_2) = 7\overbrace{\|\vec{r}_2\|}^{1=}{}^2 = 7 > 0$$

正負の両方の値をとる.

2次形式の符号.

$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ 対称.

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = (\lambda - a)(\lambda - b) - c^2$$

$$= \lambda^2 - (a+b)\lambda + ab - c^2$$

① 判別式 $D = (a+b)^2 - 4(ab - c^2)$

$$= (a-b)^2 + 4c^2 \geqq 0.$$

$\Updownarrow$

$a = b,\ c = 0$

$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI_2.$

$\alpha, \beta \geqq 0$ のとき.

$\alpha + \beta = 0 \Longrightarrow \alpha = \beta = 0$

Aの固有値は $\in \mathbb{R}$.

$\alpha, \beta \in \mathbb{R}$ $\alpha \neq \beta$ と考える.

② $\alpha \neq \beta$

$A\vec{v}_1 = \alpha \vec{v}_1,\ A\vec{v}_2 = \beta \vec{v}_2 \Longrightarrow (\vec{v}_1, \vec{v}_2) = 0.$

証明済

注)

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$

$= (\lambda - i)(\lambda + i)$

実行列では虚数の固有値ということあり.

③ $^{\exists}R$ 回転行列 $R^{-1}AR = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ とできる.

$\begin{pmatrix} x \\ y \end{pmatrix} = R\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ とすると

$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \alpha\xi^2 + \beta\eta^2$. となる.

$$= \left(R^{-1}A\begin{pmatrix} x \\ y \end{pmatrix}, R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\underbrace{R^{-1}AR}_{=\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right)$$

↑ R^{-1}回転

$$= \alpha\xi^2 + \beta\eta^2$$

④ 2次形式の符号

$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$

$$ax^2 + 2cxy + by^2$$

$$\parallel$$

(i) $\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) > 0 \quad \left(\begin{pmatrix} x \\ y \end{pmatrix} \neq 0\right)$ ($<$)

$\Updownarrow$

(ii) $a > 0$ ($<$), $|A| > 0$

$\Updownarrow$

(iii) $\alpha, \beta > 0$ $(\alpha, \beta < 0)$

(iii) $\Rightarrow$ (i). $\alpha, \beta > 0$ とする.

$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = \underset{\overset{\vee\!\!|}{0}}{\alpha\xi^2} + \underset{\overset{\vee\!\!|}{0}}{\beta\eta^2} \geqq 0.$$

$p, q \geqq 0$ のとき

$p + q = 0$

$\Rightarrow p = q = 0$

$\Leftrightarrow \alpha\xi^2 = \beta\eta^2 = 0$

$\Leftrightarrow \xi = \eta = 0$

$\Leftrightarrow \begin{pmatrix}x\\y\end{pmatrix} = R\begin{pmatrix}0\\0\end{pmatrix} = \vec{0}$

$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}\xi\\\eta\end{pmatrix}$

$$\boxed{\begin{pmatrix}x\\y\end{pmatrix} \neq \vec{0} \Rightarrow \left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) > 0}$$

(i) $\Rightarrow$ (iii)

$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) > 0 \quad \left(\begin{pmatrix}x\\y\end{pmatrix} \neq \vec{0}\right)$ とする.

$R^{-1}AR = \begin{pmatrix}\alpha & 0\\0 & \beta\end{pmatrix} \longrightarrow AR = R\begin{pmatrix}\alpha & 0\\0 & \beta\end{pmatrix} \rightarrow (A\vec{r_1}\ A\vec{r_2}) = (\alpha\vec{r_1}\ \beta\vec{r_2})$

$A\vec{r_1} = \alpha\vec{r_1}, \quad A\vec{r_2} = \beta\vec{r_2}$

$= (\vec{r_1}\ \vec{r_2})\begin{pmatrix}\alpha & 0\\0 & \beta\end{pmatrix}$

R : 回転のつもり.

$0 < (A\vec{r_1}, \vec{r_1}) = (\alpha\vec{r_1}, \vec{r_1}) = \alpha\|\vec{r_1}\|^2 = \alpha.$

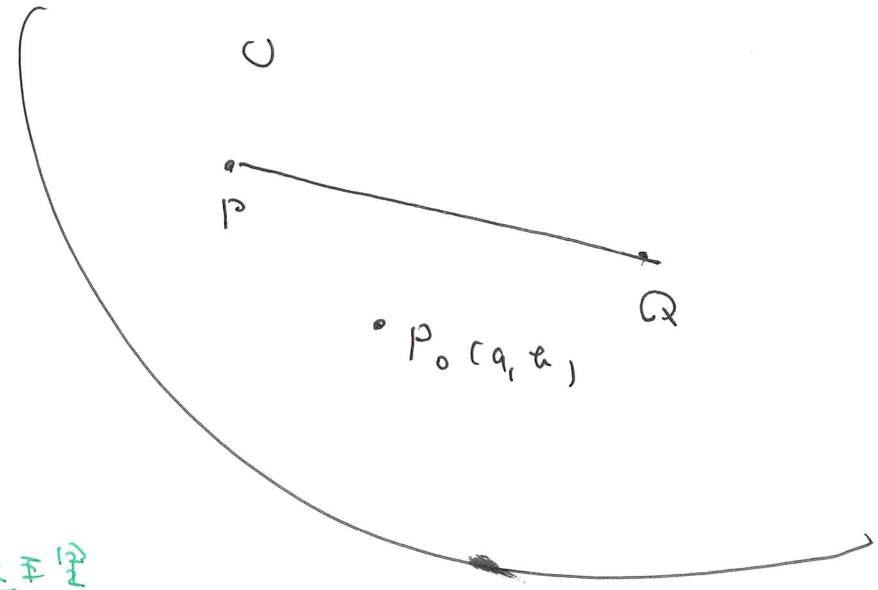
$\vec{r_1} \neq 0$

$0 < (A\vec{r_2}, \vec{r_2}) = (\beta\vec{r_2}, \vec{r_2}) = \beta\underbrace{\|\vec{r_2}\|^2}_{=1.} = \beta$

$f : U \longrightarrow \mathbb{R} \qquad U \subset \mathbb{R}^2$ 開, 凸

$P, Q \in U$

$\Longrightarrow \overline{PQ} \subset U$



1) $f_x(P_0) = f_y(P_0) = 0$

2) $f_{xx}(P) > 0 \quad (P \in U)$

3) $\begin{vmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P) \end{vmatrix} > 0$

Youngの定理

$f_{xy} = f_{yx}.$

$\Longrightarrow \quad f(P_0) < f(P) \quad (P \in U,\ P_0 \neq P)$

$H(f)(P) = \begin{pmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P) \end{pmatrix}$

$\Longleftrightarrow$

$(H(f)(P)\vec{v}, \vec{v}) > 0 \quad (\vec{v} \neq \vec{0})$

$\alpha\beta < 0 \iff ((\alpha > 0,\ \beta < 0) \text{ or } (\alpha < 0,\ \beta > 0))$

$\Updownarrow$

$|A| = ab - c^2 < 0$

$\shortparallel$

$\begin{vmatrix} a & c \\ c & b \end{vmatrix}$

<u>定理</u>　A：対称行列のとき．

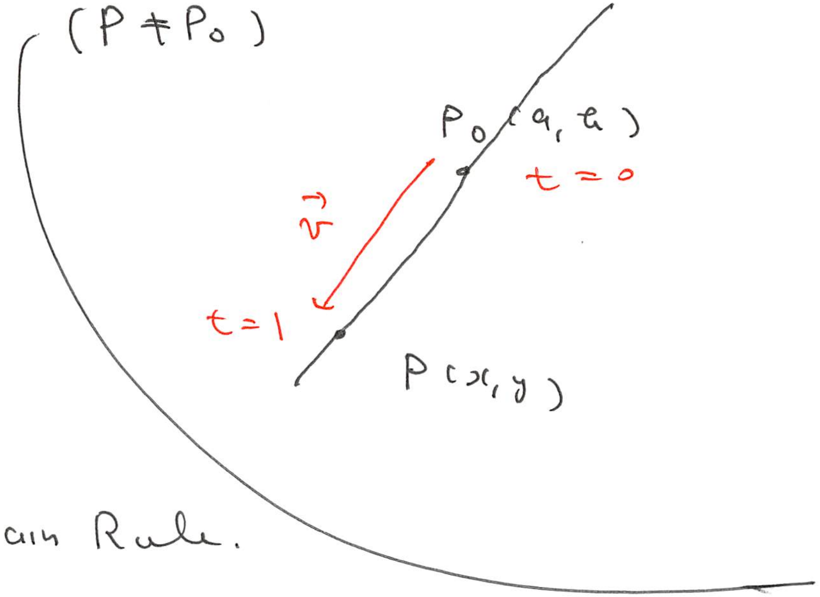
$\alpha\beta < 0 \iff |A| < 0$

$\Phi_A(\lambda) = \lambda^2 - (a+b)\lambda + ab - c^2$　　$\lambda = 0$ とすると

$= (\lambda - \alpha)(\lambda - \beta)$　　$ab - c^2 = \alpha\beta$.

このとき　$\boxed{\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) \text{は正の符号も負の符号もとりうる}}$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \overrightarrow{P_0 P} = \begin{pmatrix} x-a \\ y-b \end{pmatrix} \neq \vec{0}$$



$$F(t) = f(a + v_1 t,\ b + v_2 t)$$

と考える.

Chain Ruleを考える

$$F'(t) = f_x(\ \ )\cdot v_1 + f_y(\ \ )\cdot v_2$$

$$= \left(\nabla(f)(x(t), y(t)),\ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$$

Chain Rule.

$$\boxed{\begin{aligned} F(t) &= f(x(t), y(t)) \\ F'(t) &= f_x(x(t), y(t))\, x'(t) \\ &\quad + f_y(x(t), y(t))\, y'(t) \end{aligned}}$$

$$F''(t) = v_1 \left(f_{xx}(\ \ )\cdot v_1 + f_{xy}(\ \ )\cdot v_2 \right) + v_2 \left(f_{yx}(\ \ )\cdot v_1 + f_{yy}(\ \ )\, v_2 \right)$$

$$f_{xx} = (f_x)_x$$

$$f_{xy} = (f_x)_y.$$

$$\parallel$$

$$f_{yx} = (f_y)_x.$$

$$= f_{xx}(\ \ )\, v_1^2 + 2 f_{xy}(\ \ )\, v_1 v_2 + f_{yy}(\ \ )\, v_2^2$$

$$= \left(H(f)(\ ) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$$

$$= \left(H(f)(\ ) \vec{v}, \vec{v} \right) > 0 \quad \Leftarrow \quad 2) \ 3)$$

$$\boxed{F''(t) > 0}$$

$$F'(0) = f_x \overset{0}{\overset{\shortparallel}{(a, b)}} \cdot v_1 + f_y \overset{0}{\overset{\shortparallel}{(a, b)}} \cdot v_2 = 0$$

続きは来週.

$$z = (x^2 + y^2)^2 - 2(x^2 - y^2)$$

の停留点を求める.

f_{xx}, f_{xy}, f_{yy} を計算.