2016/08/10
6th Lecture

$y = \dfrac{\log x}{x^2}$ の増減表（省略）

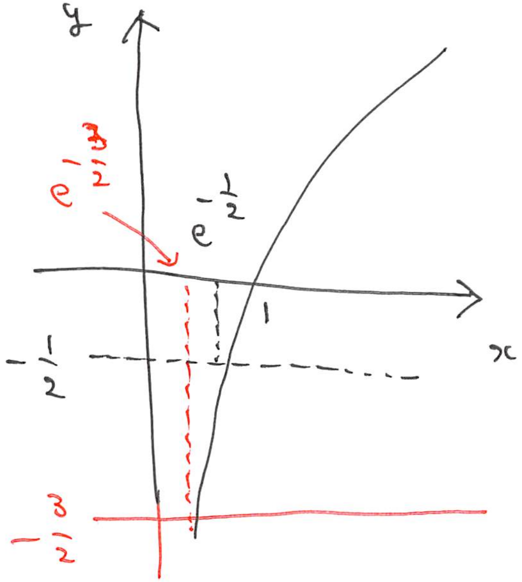
$y = \dfrac{x}{1+x^2}$ ＿＿＿＿＿＿ （〃）

増減表

(1)

$y = x^2 \log x$ 　真数条件 $x > 0$

$y = \log x$



$$y' = (x^2)' \log x + x^2 \cdot (\log x)'$$
$$= 2x \log x + x^2 \cdot \frac{1}{x}$$
$$= 2x \left(\log x + \frac{1}{2}\right)$$
$$y'' = (2x)'\left(\log x + \frac{1}{2}\right) + 2x \cdot \left(\log x + \frac{1}{2}\right)'$$
$$= 2\left(\log x + \frac{1}{2}\right) + 2x \cdot \frac{1}{x}$$
$$= 2\left(\log x + \frac{3}{2}\right)$$
$$y' \gtreqless 0 \iff \log x \gtreqless -\frac{1}{2} \iff x \gtreqless e^{-\frac{1}{2}}$$
$$y'' \gtreqless 0 \iff \log x \gtreqless -\frac{3}{2} \iff x \gtreqless e^{-\frac{3}{2}}$$

x	(0)		$e^{-\frac{3}{2}}$		$e^{-\frac{1}{2}}$	
y'	/	$-$	$-$	$-$	0	$+$
y''	/	$-$	0	$+$	$+$	$+$
y	(0)	$\searrow$		$\searrow$	$-\frac{1}{2e}$	$\nearrow$

$$e^{-3}\left(-\frac{3}{2}\right) = -\frac{3}{2e^3} \qquad e^{-1}\left(-\frac{1}{2}\right) = -\frac{1}{2e}$$

$x = e^{-s}$

$\Updownarrow$

$x \to +0 \qquad s = -\log x \to +\infty$

$$x^2 \log x = e^{-2s}(-s) = -\frac{1}{2}\cdot\frac{2s}{e^{2s}} \longrightarrow -\frac{1}{2}\cdot 0$$

$$\boxed{\frac{t}{e^t} \to 0 \quad (t \to +\infty}$$

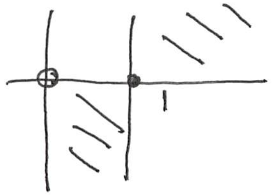
$x \to +\infty$ のとき.

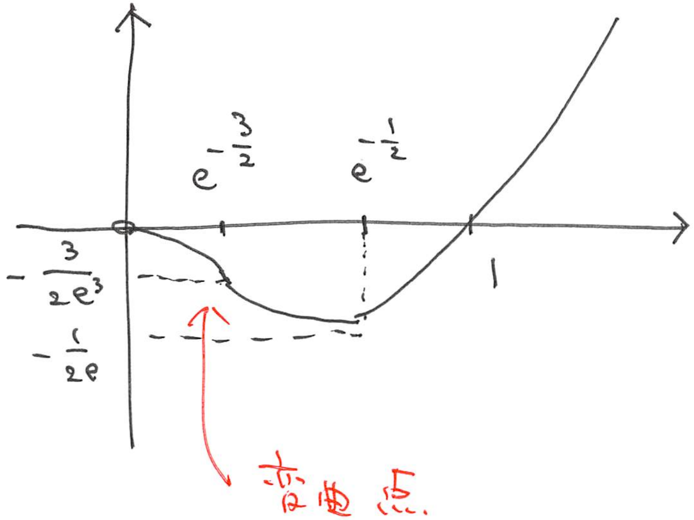
$\log x \to +\infty$

$x^2 \log x \to +\infty$

$x^2 \log x \gtreqless 0 \iff x \gtreqless 1$

$x^2 > 0$ $(x > 0)$





$e^{-\frac{3}{2}}$
$e^{-\frac{1}{2}}$
1
$-\frac{3}{2e^3}$
$-\frac{1}{2e}$
変曲点

(3) $y = \dfrac{\log x}{x}$

$$y' = \frac{(\log x)' x - \log x\,(x)'}{x^2}$$

$$= \frac{\frac{1}{x} \cdot x - \log x \cdot 1}{x^2}$$

$$= \frac{1 - \log x}{x^2}$$

$$y'' = \frac{(1-\log x)' x^2 - (1-\log x)(x^2)'}{x^4}$$

$$= \frac{-\frac{1}{x} \cdot x^2 - (1-\log x) \cdot 2x}{x^4}$$

$$= \frac{-1 - 2(1-\log x)}{x^3} = \frac{2\left(\log x - \frac{3}{2}\right)}{x^3}$$

$y' \gtreqless 0 \iff 1 - \log x \gtreqless 0 \iff \log x \lesseqgtr 1 \iff x \lesseqgtr e$

$$y'' \gtreqless 0 \iff \log x - \frac{3}{2} \gtreqless 0 \iff x \gtreqless e^{\frac{3}{2}}$$

$$y = \frac{\log x}{x}$$

(0)		e		$e^{\frac{3}{2}}$	
	$+$	0	$-$	$-$	$-$
	$-$	$-$	$-$	0	$+$
	↗	$\frac{1}{e}$	↘	$\frac{3}{2e\sqrt{e}}$	↘

$$\frac{\frac{3}{2}}{e^{\frac{3}{2}}} = \frac{3}{2e\sqrt{e}}$$

$x = e^t$

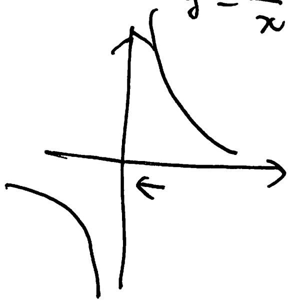
$x \to +\infty$ のとき　$t = \log x \to +\infty$

$$\frac{\log x}{x} = \frac{t}{e^t} \longrightarrow 0$$

$$\boxed{\begin{array}{c} \frac{t}{e^t} \to 0 \\ t \to +\infty \end{array}}$$

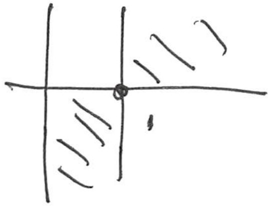
$x \to +0$ のとき　$\left.\begin{array}{l} \log x \to -\infty \\ \frac{1}{x} \to +\infty \end{array}\right\}$ $\frac{\log x}{x} \longrightarrow -\infty$

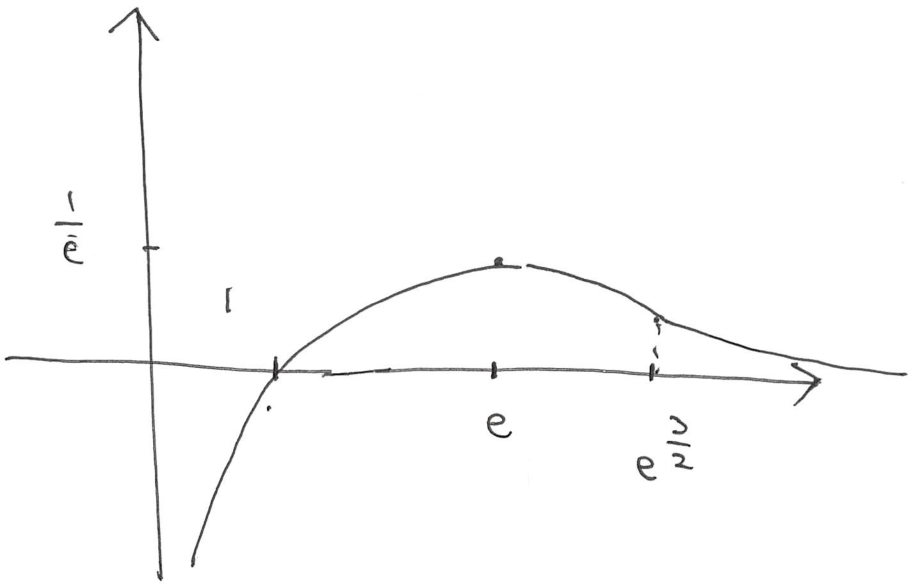
$y = \frac{1}{x}$



$$\frac{\log x}{x} \gtreqless 0 \iff x \gtreqless 1$$

$x > 0$





問　$f(x) = \log(1+x)$ のとき $f^{(n)}(x) = ?$　　　$u = 1+x,\ \dfrac{du}{dx} = 1$

$$f'(x) = \underbrace{\tfrac{1}{u}\cdot 1}_{} = \frac{1}{1+x} \qquad (\log(1+x) = \log u.)$$

$$f''(x) = -\frac{1}{u^2}\cdot 1 = -\frac{1}{(1+x)^2}$$

$$f^{(3)}(x) = \frac{2}{u^3}\cdot 1 = \frac{2!}{(1+x)^3}$$

$$f^{(4)}(x) = -\frac{2\cdot 3}{u^4} = -\frac{3!}{(1+x)^4}$$

$$f^{(5)}(x) = \frac{2\cdot 3\cdot 4}{u^5} = \frac{4!}{(1+x)^5}$$

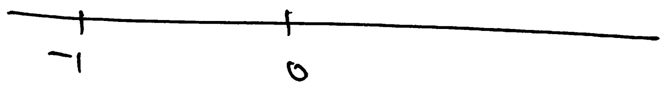
$$\boxed{0! = 1.}$$

＊　$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$$　　　$n(n-1)! = n!$

$$f^{(n+1)}(x) = (-1)^{n-1}\cdot(n-1)!\,\frac{(-n)}{(1+x)^{n+1}} = (-1)^{n+1-1}\frac{(n+1-1)!}{x^{n+1}}$$

帰納法により ＊ が常に成立する。

$f(x) = \log(1+x) \quad (x > -1)$



3階のTaylorの定理を適用すると

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3!}\frac{1}{(1+c)^3}x^3$$

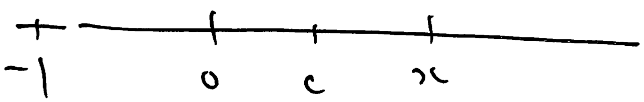
を満たす c が 0 と x の間に存在する.

$$\log(1+x) - \left(x - \frac{1}{2}x^2\right) = \frac{1}{3!}\cdot\frac{1}{(1+c)^3}x^3$$

<u>$x > 0$ のとき</u>

$1+c > 0$ より

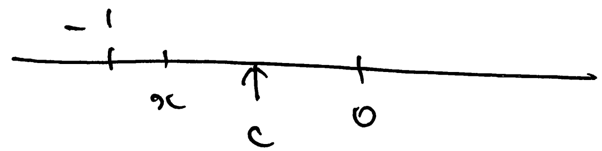
$$\frac{1}{(1+c)^3} > 0$$



よって $\frac{1}{3!}\cdot\frac{1}{(1+c)^3}x^3 > 0 \longrightarrow \log(1+x) > x - \frac{1}{2}x^2$

<u>$x < 0$ のとき.</u>

$1+c > 0$ より $\frac{1}{(1+c)^3} > 0$



$\frac{1}{3!}\frac{1}{(1+c)^3}x^3 < 0 \longrightarrow \log(1+x) < x - \frac{1}{2}x^2$

$$f(x) = \sqrt{1+x}. \qquad a = 0.$$

$$\{(1+x)^{\alpha}\}' = \alpha(1+x)^{\alpha-1}$$

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{1+x}}$$

$$f(0) = 1, \quad f'(0) = \frac{1}{2}, \quad f''(0) = -\frac{1}{4}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$f^{(3)}(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}}$$

$x \fallingdotseq a$ のとき

$$\boxed{f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f^{(3)}(c)(x-a)^3}$$

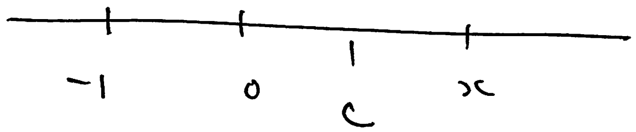
$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}(1+c)^{-\frac{5}{2}}x^3$$

$x \fallingdotseq 0$

$$\sqrt{1+x} \fallingdotseq 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

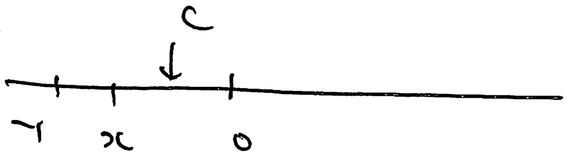
2次式 近似に.

$x > 0$



$1 + c > 0$

$x < 0$



$1 + c > 0$

$$(1+c)^{-\frac{5}{2}} > 0.$$

$x > 0$ のとき $\sqrt{1+x} > 1 + \frac{1}{2}x - \frac{1}{8}x^2$

$x < 0$ のとき $<$

$f(x) = \log(1+x)$, $a=0$ で 4階の Taylorの定理.

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f^{(3)}(x) = \frac{2}{(1+x)^3}, \quad f^{(4)}(x) = -\frac{6}{(1+x)^4}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f^{(3)}(0) = 2$$

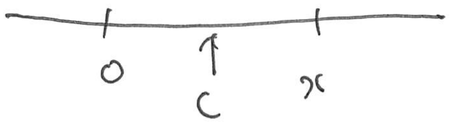
$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \frac{1}{3!} f^{(3)}(a)(x-a)^3 + \boxed{\frac{1}{4!} f^{(4)}(c)(x-a)^4}$$

$x \neq 0$

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}\frac{1}{(1+c)^4}x^4$$

をみたす c が 0 と x の間に存在

$x>0$ のとき

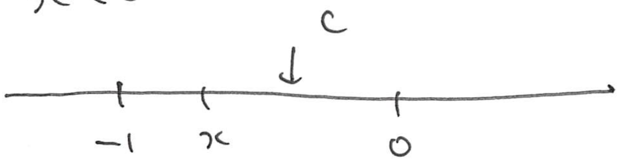


$c > 0 > -1$

$\frac{1}{(1+c)^4} > 0$ より 剰余項

$$-\frac{1}{4} \cdot \frac{1}{(1+c)^4} x^4 < 0 \quad (x \neq 0)$$

$x<0$



$c > -1$

不等式 $\log(1+x) < x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \quad (x \neq 0)$ が成立.

$y = x e^{2x}$.

$$y' = (x)' e^{2x} + x (e^{2x})'$$

$$= 1 \cdot e^{2x} + x \cdot 2 e^{2x}$$

$$= (2x+1) e^{2x} > 0$$

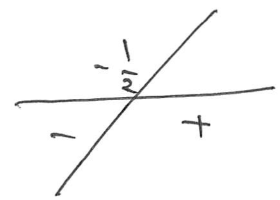
$$y'' = (2x+1)' e^{2x} + (2x+1)(e^{2x})'$$

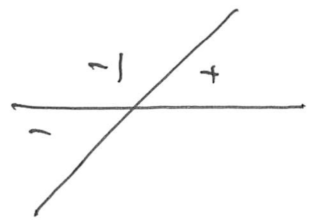
$$= 2 e^{2x} + (2x+1) \cdot 2 e^{2x}$$

$$= 4(x+1) e^{2x} > 0$$

$(e^{ax})' = a e^{ax}$

a : 定数





x		-1		$-\frac{1}{2}$	
y'	$-$	$-$	$-$	0	$+$
y''	$-$	0	$+$	$+$	$+$
y	⤵	$-\frac{1}{e^2}$	⤷	$-\frac{1}{2e}$	⤴

$-e^{-2}$ $\quad$ $-\frac{1}{2} e^{-1}$

$x \to +\infty$ のとき $2x \to +\infty$ より $e^{2x} \to +\infty$ よって $xe^{2x} \to +\infty$

$x \to -\infty$ のとき.

$\boxed{2x \to -\infty \text{ より } e^{2x} \to 0}$

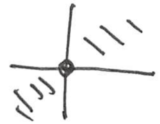
$-2x = s$ とする $s \to +\infty$

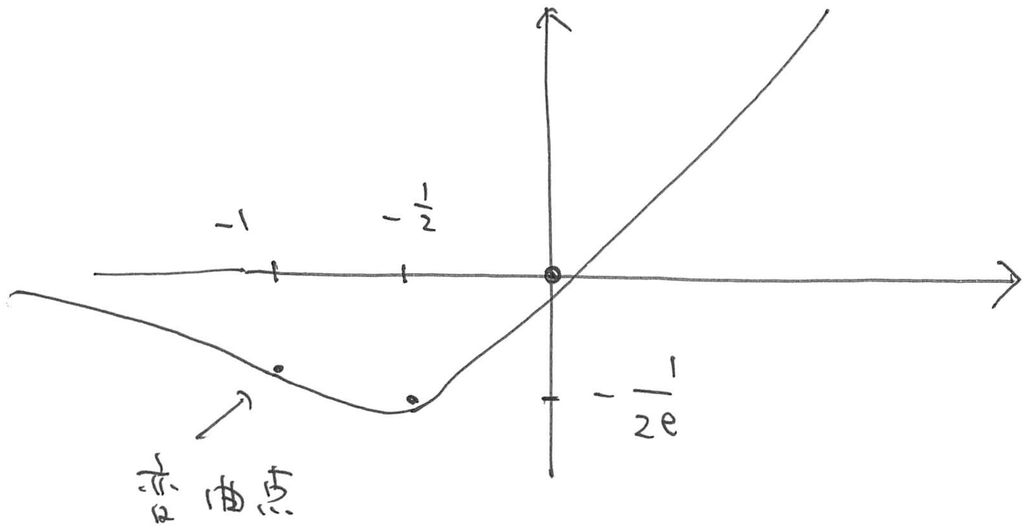
$$xe^{2x} = -\frac{1}{2} s e^{-s} = -\frac{1}{2}\frac{s}{e^{s}} \to -\frac{1}{2}\cdot 0 \quad (s \to +\infty)$$

$\bigcirc$

$\wedge$

$x e^{2x} \gtreqless 0 \iff x \gtreqless 0$





補足

$f(x) = e^x$

$x \neq 0$

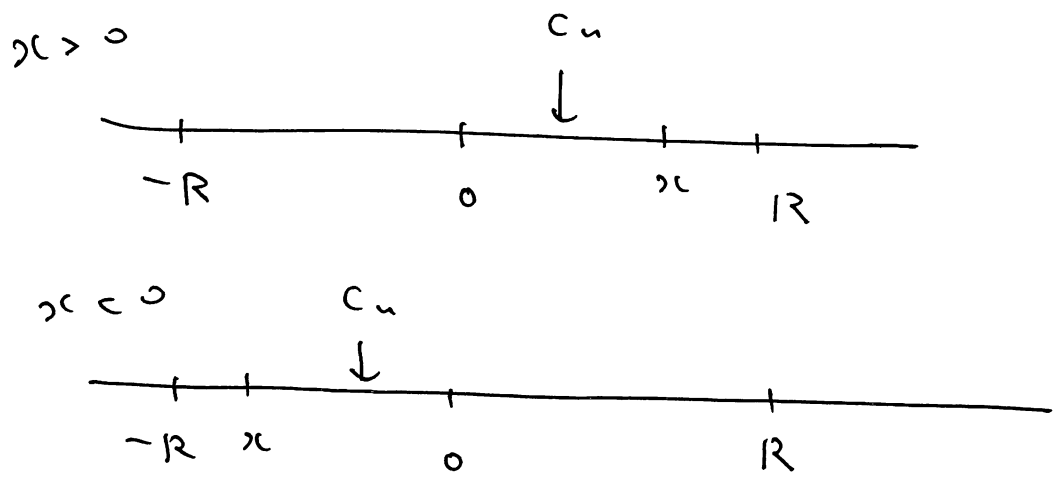
$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{k!}x^k + \cdots + \frac{1}{(n-1)!}x^{n-1} + \frac{1}{n!}e^{c_n}x^n$$

を満たす c_n が 0 と x の間に存在.

$|x| \leq R$.

$c_n \leq R$ となる

$e^{c_n} \leq e^R$



$$e^x - \sum_{k=0}^{n-1} \frac{1}{k!} x^k = \frac{1}{n!} e^{c_n} x^n \to 0$$

$$| \bigcirc | \leq \underbrace{\frac{1}{n!} R^n \cdot e^R}$$

$$\downarrow \quad n \to +\infty$$

$$0$$

$$e^x - \sum_{k=0}^{+\infty} \frac{1}{k!} x^k = 0$$

$$-\frac{1}{n!} R^n e^R \leq \bigcirc \leq \frac{1}{n!} R^n e^R$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$-0 = 0 \qquad 0 \qquad 0$$

$$\boxed{\begin{array}{l} R > 0 \\ \dfrac{R^n}{n!} \longrightarrow 0 \quad (n \to +\infty) \end{array}}$$

129p.

$$e^x = 1 + x + \frac{1}{2} x^2 + \cdots$$

e^x の $x = 0$ における Taylor 展開.

$$\log(1+x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + - \cdots$$

$|x| < 1$

131p.

$0 < R < N$ を満たす整数 N をとる　　　$n > N$

$$0 < \frac{R^n}{n!} = \frac{R}{1} \cdot \frac{R}{2} \cdot \cdots \cdot \frac{R}{N} \cdot \frac{R}{(N+1)} \cdot \cdots \cdot \frac{R}{n} \to 0$$

$$0 < \frac{R}{N} < 1$$

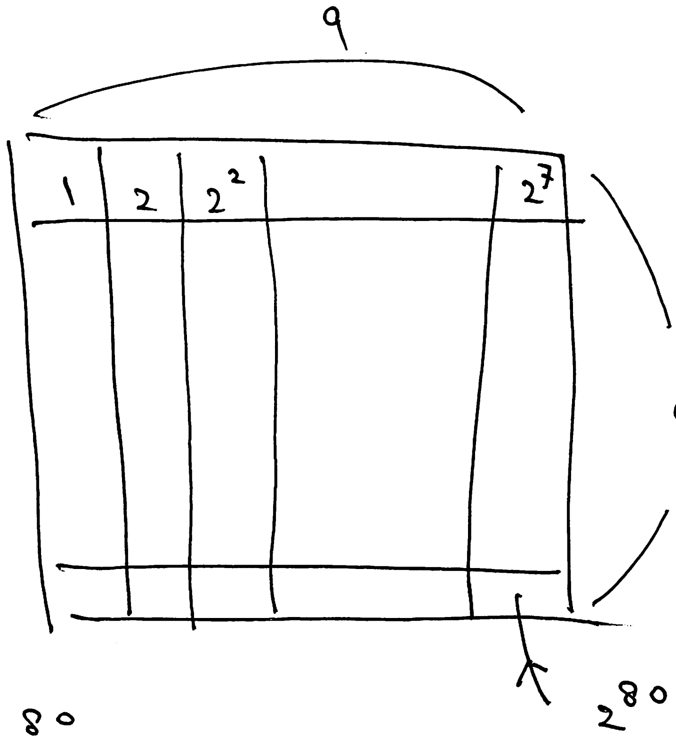
$$\frac{R}{(N+1)} < \frac{R}{N}, \qquad \frac{R}{n} < \frac{R}{N}$$

$$< \frac{R^N}{N!} \cdot \left(\frac{R}{N}\right)^{n-N} = \frac{R^N}{N!} \cdot \left(\frac{R}{N}\right)^{-N} \left(\frac{R}{N}\right)^n$$

$$\left(\frac{R}{N}\right)^n \to 0$$

$$0$$

曾呂利新左衛門



81 マス.

$2^{10} = 1024 \doteqdot 10^3$

$$\frac{2^{81} - 1}{2 - 1} \doteqdot 2 \cdot 2^{80}$$

$$\doteqdot 2 \cdot 10^{24}.$$

$R > 1$

$$\frac{n^k}{R^n} \to 0.$$