2016/08/08
4th Lecture

$f'(x)$ を求めよ

(1) $y = \dfrac{x}{x^2 + x + 1}$

$$y' = \frac{(x)'(x^2+x+1) - x(x^2+x+1)'}{(x^2+x+1)^2}$$

$$= \frac{1\cdot(x^2+x+1) - x(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{-x^2+1}{(x^2+x+1)^2}$$

(2) $y = \dfrac{1}{(3x+1)^3} = \dfrac{1}{u^3}$ $\quad u = 3x+1$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = -\frac{3}{u^4}\cdot 3 = -\frac{9}{(3x+1)^4}$$

(3) $y = (1-2x)^5 = u^5$ と置く $u = 1-2x$.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 5u^4(-2) = -10(1-2x)^4$$

(4) $y = \dfrac{1}{(3x-2)^5} = \dfrac{1}{u^5}$ $\qquad u = 3x - 2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{5}{u^6} \cdot 3 = -\frac{15}{(3x-2)^6}$$

(5) $y = \left(\dfrac{x-1}{x}\right)^5 = u^5$ $\qquad u = \dfrac{x-1}{x} = 1 - \dfrac{1}{x}$ $\qquad \boxed{\left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2}}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4 \cdot \left(\frac{1}{x^2}\right) = 5\left(\frac{x-1}{x}\right)^4 \cdot \frac{1}{x^2}$$

(6) $y = \dfrac{1}{\sqrt{1+x+x^2}} = \dfrac{1}{\sqrt{u}}$ $\qquad u = 1 + x + x^2$

$$\left(u^{-\frac{1}{2}}\right)' = -\frac{1}{2} u^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{1}{2} \cdot \frac{1}{u\sqrt{u}} \cdot (2x+1) = -\frac{2x+1}{2(1+x+x^2)\sqrt{1+x+x^2}}$$

I (1) $f(x) = \dfrac{x^2}{1+x^2}$ $\quad f'(x) = \dfrac{(x^2)'(1+x^2) - x^2(1+x^2)'}{(1+x^2)^2}$

$$= \frac{2x(1+x^2) - x^2 \cdot 2x}{(1+x^2)^2}$$

$$= \frac{2x}{(1+x^2)^2}$$

$(1+x^2)^2 > 0$ だから

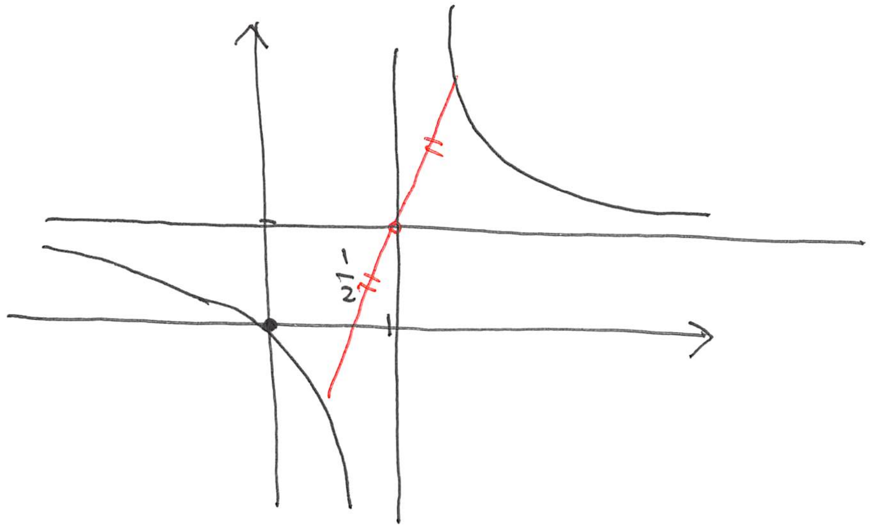
$f'(x) \gtreqless 0 \iff x \gtreqless 0$

x		0	
y'	$-$	0	$+$
y	$\searrow$	0	$\nearrow$

(2) $y = \dfrac{x}{2x-1}$ $\quad y' = \dfrac{(x)'(2x-1) - x(2x-1)'}{(2x-1)^2}$

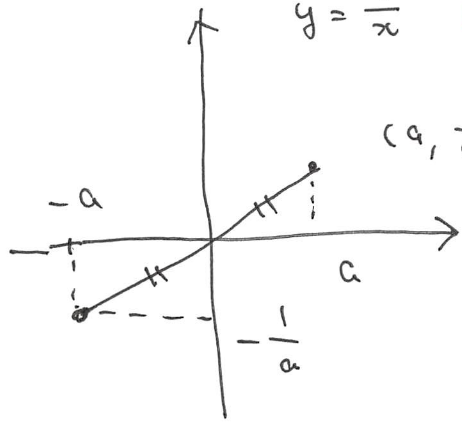
$$= \frac{1\cdot(2x-1) - x\cdot 2}{(2x-1)^2} = -\frac{1}{(2x-1)^2}$$

$$y = \frac{x}{2(x-\frac{1}{2})} = \frac{x-\frac{1}{2}+\frac{1}{2}}{2(x-\frac{1}{2})} = \frac{1}{2} + \frac{1}{2(x-\frac{1}{2})}$$



$y = \frac{1}{2}\cdot\frac{1}{x}$

$y = \frac{1}{x}$ は原点について点対称.



(3) $f(x) = \dfrac{x}{1+x+x^2}$ 省略.

(4) $f(x) = (x-1)\sqrt{x}$. $(fg)' = f'g + fg'$

$f'(x) = (x-1)'\sqrt{x} + (x-1)(\sqrt{x})'$

$= 1 \cdot \sqrt{x} + (x-1)\dfrac{1}{2} \cdot \dfrac{1}{\sqrt{x}}$

$= \dfrac{\sqrt{x} \cdot 2\sqrt{x} + (x-1)}{2\sqrt{x}} = \dfrac{3x-1}{2\sqrt{x}}$.

$2\sqrt{x} > 0$

$f'(x) \gtreqless 0 \iff 3x-1 \gtreqless 0$

$\iff x \gtreqless \dfrac{1}{3}$

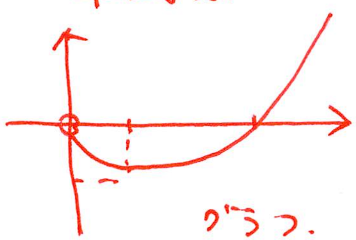
x	(0)		$\frac{1}{3}$		
y'	/	$-$	0	$+$	
y	(0)	↘		↗	

$-\dfrac{2}{3} \cdot \dfrac{1}{\sqrt{3}} = -\dfrac{2}{3\sqrt{3}}$

さらに $f''(x) = \dfrac{1}{2} \cdot \dfrac{3\sqrt{x} - (3x-1)\frac{1}{2\sqrt{x}}}{x} = \dfrac{2 \cdot 3x - (3x-1)}{4x\sqrt{x}}$.

$= \dfrac{3x+1}{4x\sqrt{x}} > 0$

より $y = f(x)$ は下に凸



グラフ.

II (1) $f(x) = \left(\dfrac{x-1}{x}\right)^4$ 商 (1) $\frac{2}{6}$.

$$f'(x) = 4\left(\frac{x-1}{x}\right)^3 \left\{\frac{x-1}{x}\right\}'$$

$$= 4\left(\frac{x-1}{x}\right)^3 \frac{(x-1)'x-(x)'(x-1)}{x^2}$$

$$= 4\,\frac{(x-1)^3}{x^5}$$

(2) y "$f(x) = (1+x^2)^6 = u^6 \quad (u = 1+x^2)$

$$= \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx} = 6u^5 \cdot 2x = 12x(1+x^2)^5$$

(3) $y = f(x) = \sqrt{3x+1} = \sqrt{u} \qquad u = 3x+1$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{2}\frac{1}{\sqrt{u}}\cdot 3 = \frac{3}{2\sqrt{3x+1}}$$

III

$$\begin{cases} a_{n+2} - \overset{1+3}{4}\, a_{n+1} + \overset{1\cdot 3}{3}\, a_n = 0 \\ a_0 = c_0,\quad a_1 = c_1 \end{cases}$$

特性方程式は

$$\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0 \iff \lambda = 1, 3.$$

$$\begin{cases} a_{n+2} - a_{n+1} = 3(a_{n+1} - a_n) \\ a_{n+2} - 3a_{n+1} = a_{n+1} - 3a_n \end{cases}$$

より $\{a_{n+1} - a_n\}$ は公比3の等比数列

$\{a_{n+1} - 3a_n\}$ は定数列

$$\begin{cases} a_{n+1} - a_n = 3^n (a_1 - a_0) \\ a_{n+1} - 3a_n = a_1 - 3a_0 \end{cases}$$

$$-)\quad \overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$2a_n = 3^n (a_1 - a_0) - (a_1 - 3a_0)$$

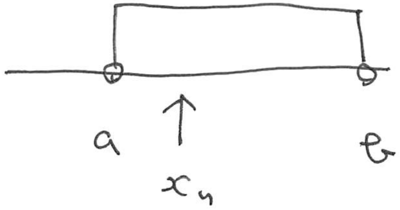
よって

$$a_n = 3^n \frac{a_1 - a_0}{2} - \frac{a_1 - 3a_0}{2}$$

平均値の定理，指数関数・対数関数.

右極限.

左



$$f : (a, b) \longrightarrow \mathbb{R}$$

$x \to a+0$ のとき　(右極限.)

$$f(x) \longrightarrow A$$

$\Longleftrightarrow$ $a < x_n < b$, $x_n \longrightarrow a$ $(n \to +\infty)$

を満たす $\{x_n\}$ に対して

$$f(x_n) \longrightarrow A.$$

$x \to b-0$　(左極限.)

のとき

$$f(x) \to B$$

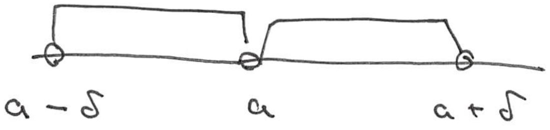
$\Longleftrightarrow$ $a < x_n < b$, $x_n \longrightarrow b$ $(n \to +\infty)$

を満たす $\{x_n\}$ に対して

$$f(x_n) \longrightarrow B.$$

$\delta > 0$

$f : (a-\delta, a) \cup (a, a+\delta) \to \mathbb{R}$



定理 $x \to a$ のとき $f(x) \to A$ ← 両側極限.

(定理 2.6 59p.) $\Longrightarrow$ $x \to a+0$ のとき $f(x) \to A$

$a-0$ $\qquad$ A

$x_n = (-1)^n \frac{1}{n} + a \to a$

$$-\frac{1}{n} \le \underbrace{x_n - a}_{=(-1)^n \frac{1}{n}} \le \frac{1}{n}, \quad \underbrace{x_n - a + a}_{=x_n} \to 0 + a = a.$$

$-\frac{1}{n} \to 0$, $\frac{1}{n} \to 0$, $(-1)^n \frac{1}{n} \to 0$

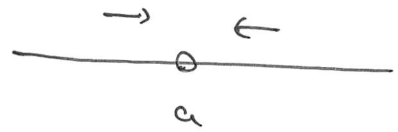
by はさみうち

→ 但し教科書の証明はこの逆を重点としている.

定理

$x \to a+0$ のとき $f(x) \to A_1$

$x \to a-0$ のとき $f(x) \to A_2$



$$A_1 = A_2$$

$$\Longrightarrow \quad x \to a \text{ のとき } f(x) \to A_1 (= A_2)$$

(証明は60p.にある) 追って 数列を用いて関数の極限を定義していたが、これを関数の値を用いて定義し直す。

例えば $x \to a$ のとき $f(x) \to A$

$\Longleftrightarrow$ $\forall \varepsilon > 0$ に対して $\exists \delta > 0$ が存在して

$$A - \varepsilon < f(x) < A + \varepsilon \quad \left(\begin{array}{c} a - \delta < x < a + \delta \\ x \neq 0 \end{array} \right)$$

と書き換える。

本質的なのは次の定理である。

$f(x) \to A > 0 \quad (x \to a)$

$\Longrightarrow \exists \delta > 0$

$$f(x) > 0 \quad \left(\begin{array}{c} x \neq 0 \\ a - \delta < x < a + \delta \end{array} \right)$$

$a_n = \left(1+\frac{1}{n}\right)^n$ は収束して $a_n \to e \quad (n \to +\infty)$

$\wedge|$

3

$$e = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n$$

$\cdots < a_n < a_{n+1} < \cdots$

$\left(1+\frac{1}{t}\right)^t \to e \quad (t \to +\infty)$ ← t の関数として。 (3.9)

$t_n \to +\infty$ のとき

$$\left(1+\frac{1}{t_n}\right)^{t_n} \to e \quad (n \to +\infty)$$

$t \to -\infty$ のとき。 $\left(1+\frac{1}{t}\right)^t \to e$ (3.10)

$t = -s$ とする。 $t \to -\infty$ のとき $s \to +\infty$

$s-1+1$

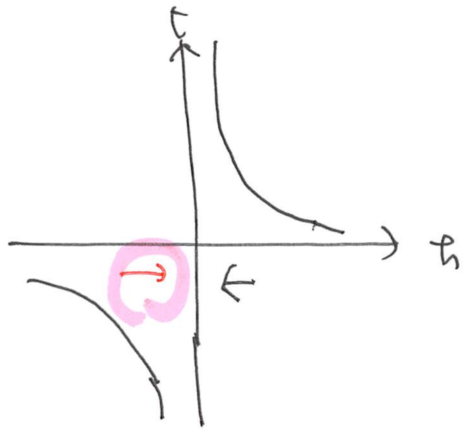
$$\left(1+\frac{1}{t}\right)^t = \left(1-\frac{1}{s}\right)^{-s} = \left(\frac{s-1}{s}\right)^{-s} = \left(\frac{s}{s-1}\right)^{s}$$

$$= \left(1+\frac{1}{s-1}\right)^{s-1+1} = \left(1+\frac{1}{s-1}\right)^{s-1}\left(1+\frac{1}{s-1}\right)$$

$\longrightarrow e \cdot 1 = e$

(3.9) $\quad \left(1+\frac{1}{t}\right)^t \longrightarrow e \quad (t \to +\infty)$

$\left(h = \frac{1}{t} > 0 \text{ より } h \to +0. \right)$



$h \to +0$ のとき $t = \frac{1}{h} \longrightarrow +\infty$

$(1+h)^{\frac{1}{h}} = \left(1+\frac{1}{t}\right)^t \longrightarrow e$

$\longrightarrow$ (3.9)' $\quad (1+h)^{\frac{1}{h}} \longrightarrow e \quad (h \to +0)$

(3.10) $\quad \left(1+\frac{1}{t}\right)^t \to e \quad (t \to -\infty)$

$h \to -0$ のとき $t = \frac{1}{h} \longrightarrow -\infty.$

$(1+h)^{\frac{1}{h}} = \left(1+\frac{1}{t}\right)^t \longrightarrow e$

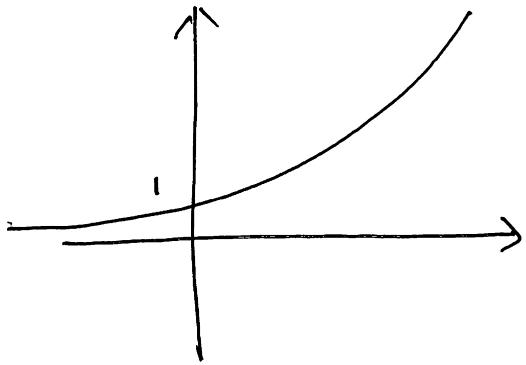
$\to$ (3.10)' $\quad (1+h)^{\frac{1}{h}} \to e \quad \boxed{(h \to -0)}$

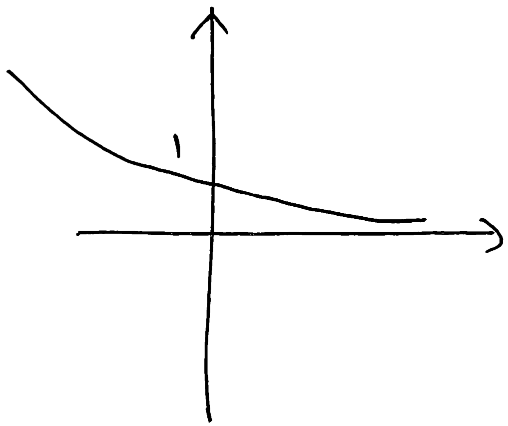
$$(1+h)^{\frac{1}{h}} \to e \quad (h \to 0)$$

55p.

指数関数の連続性　　$a \neq 1,\ a > 0$　　$0 < a < 1$

$y = a^x$





$x \to x_0$ のとき

$a^x \to a^{x_0}$.　← 指数関数の連続性

56p, 57p.

対数関数の連続性 $a > 0,\ a \neq 1$

$$x = a^y \iff y = \log_a x .$$

中間値の定理.

$$\log_a x \to \log_a x_0 \quad (x \to x_0)$$

$h \to 0$ のとき $(1+h)^{\frac{1}{h}} \longrightarrow e.$

対数関数の連続性.

$$\log (1+h)^{\frac{1}{h}} \longrightarrow \log e$$

$$\log (1+h)^{\frac{1}{h}} = \frac{\log(1+h)}{h}, \quad \log e = 1$$

$\log * = \log_e *$

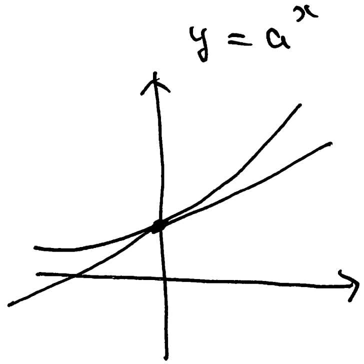
$$\boxed{\frac{\log(1+h)}{h} \longrightarrow 1 \quad (h \to 0)}$$

$x = \log(1+h)$ とおく.　$e^{x} = 1 + h$, $h = e^{x} - 1$.

$x \to 0$ のとき　$e^{x} \to e^{0} = 1$, $h = e^{x} - 1 \to 1 - 1 = 0$

指数関数の連続性

$$\frac{x}{e^{x}-1} = \frac{\log(1+h)}{h} \longrightarrow 1 \quad (x \to 0)$$



$$\frac{e^{x}-1}{x} \longrightarrow \frac{1}{1} = 1 \quad (x \to 0)$$

$$\boxed{\frac{e^{x}-1}{x} \longrightarrow 1 \quad (x \to 0)}$$

指数関数

$$\frac{e^x - e^{x_0}}{x - x_0} = \frac{e^{x_0+h} - e^{x_0}}{h}$$

$h = x - x_0$ とおく.

$h \to 0$

$= e^{x_0} \cdot e^h$

$x \to x_0$.

$$= e^{x_0} \cdot \frac{e^h - 1}{h} \longrightarrow e^{x_0} \cdot 1 = e^{x_0}$$

$$\boxed{(e^x)' = e^x.}$$

対数関数 $x, x_0 > 0$ $x - x_0 = h$

$$\frac{\log x - \log x_0}{x - x_0} = \frac{\log(x_0 + h) - \log x_0}{h}$$

$$= \frac{1}{h} \log\left(1 + \frac{h}{x_0}\right)$$

$$= \frac{1}{x_0} \cdot \boxed{\frac{1}{\frac{h}{x_0}} \log\left(1 + \frac{h}{x_0}\right)} \to 1$$

$$\boxed{\frac{\log(1+h)}{h} \to 1 \quad h \to 0}$$

$\frac{h}{x_0} \to 0$ より

$$\longrightarrow \frac{1}{x_0} \cdot 1 = \frac{1}{x_0}$$

$$\boxed{(\log x)' = \frac{1}{x}}$$

演習　次の $f(x)$ に対して $f'(x)$ を求めよ

(1) e^{2x} "u"　(2) e^{-x}　(3) $e^{-\frac{1}{2}x^2}$　(4) xe^{x}

(5) $\dfrac{1}{1+e^{x}}$　(6) $\boxed{\cancel{x\log x}}$　(7) $x^2\log x$　(8) $\dfrac{\log x}{x}$

(9) $\log(1+x^2)$

$y = e^{2x} = e^{u} \quad u = 2x.$ $\quad (e^{x})' = e^{x}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^{u} \cdot 2 = 2e^{2x}.$$

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a：定数 $\quad u = ax$

$y = e^{ax} = e^{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^{u} \cdot a = a e^{ax}$$

<u>公式</u> a，定数のとき $\quad (e^{ax})' = a e^{ax}$

$x > 0 \quad \alpha \in \mathbb{R} \quad y = x^{\alpha} = \left(e^{\log x}\right)^{\alpha}$ $\qquad x = e^{\log x}.$

$= e^{\alpha \log x} = e^{u}$ $\qquad u = \alpha \log x.$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^{u} \cdot \alpha \frac{1}{x} = x^{\alpha} \cdot \alpha \frac{1}{x}$$

$$= \alpha x^{\alpha - 1}$$

$$\boxed{\left(x^{\alpha}\right)' = \alpha x^{\alpha-1}}$$

$\sqrt[3]{-27} = -3$

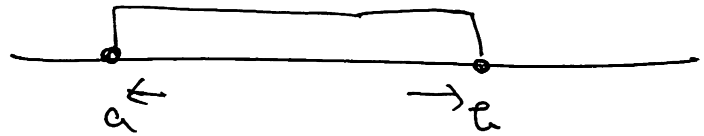
$(-3)^3 = -27.$

平均値の定理

$$f : [a, b] \longrightarrow \mathbb{R}$$

$$= \{ x \in \mathbb{R} ;\ a \leq x \leq b \}$$

内点



Rolleの定理

① f は (a, b) の各点で微分可能

② $[a, b]$ の各点で連続

(i) $t_0 \in (a, b)$ のとき

$$t \to t_0 \Rightarrow f(t) \to f(t_0)$$

(ii) $t \to a+0 \Rightarrow f(t) \to f(a)$

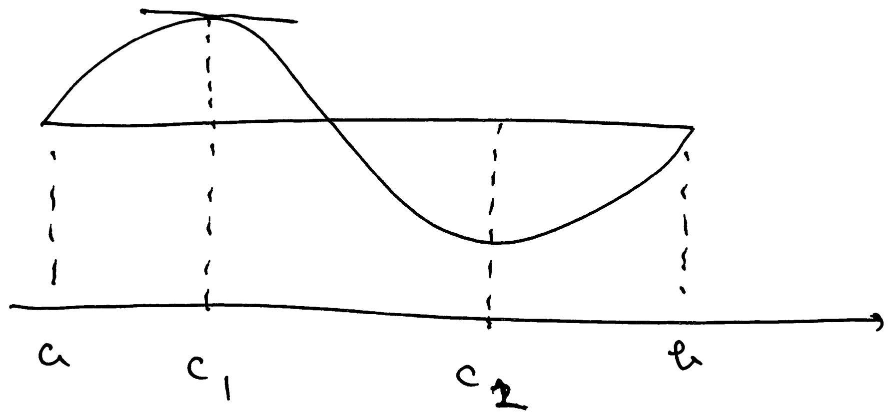
(iii) $t \to b-0 \Rightarrow f(t) \to f(b)$

定理　$f(a) = f(b)$ とすると

$f'(c) = 0$ を満たす

$c \in (a, b)$

が存在する.



61ページ　(最大値の定理)

定理　$g : [a, b] \longrightarrow \mathbb{R}$　$[a, b]$ の各点で連続.

$\Longrightarrow$ g には最大値・最小値が存在する.

M 最大値 とは 1) ある $x_0 \in [a, b]$ に対して

$$M = g(x_0)$$

2) $g(x) \leq M \quad (x \in [a, b])$

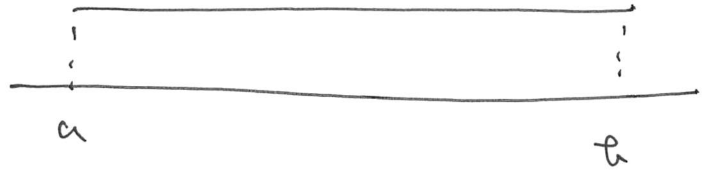
m 最小値 とは 1) ある $x_1 \in [a, b]$ に対して $m = g(x_1)$

2) $m \leq g(x) \quad (x \in [a, b])$

(i) $f(x) \equiv f(a) = f(b)$ 定数関数

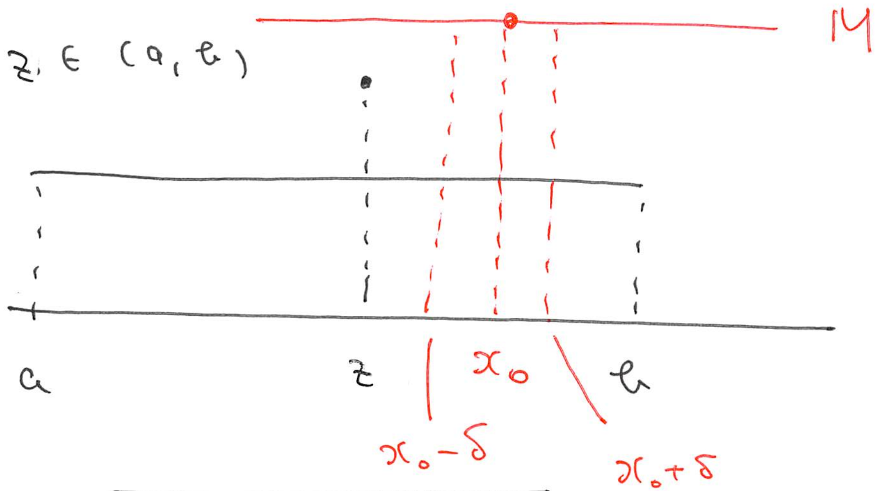
$\uparrow$

常に.



$f'(x) \equiv 0$ より OK

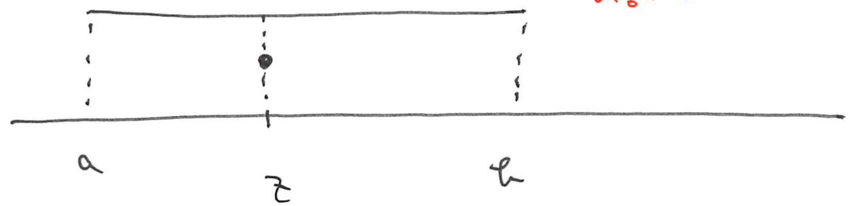
(ii) (i) ではないとする. ある $z \in (a, b)$



(あ) $f(z) > f(a) = f(b)$

OR

(い) $f(z) < f(a) = f(b)$



最大値. $M = f(x_0)$

$m = f(x_1)$

(あ)

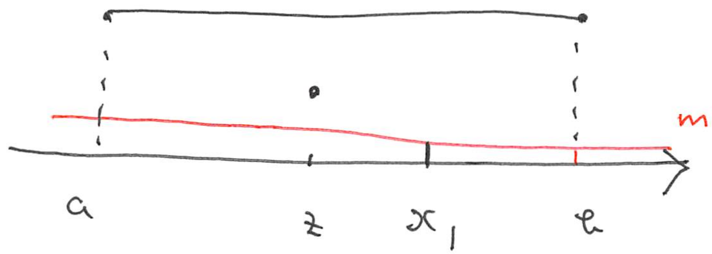
$$\underset{f(x_0)}{\overset{}{M}} \geqq f(z) > f(a) = f(b) \rightsquigarrow a < x_0 < b.$$

$\rightarrow$ $f(x) \leqq f(x_0)$ $(x_0 - \delta < x < x_0 + \delta)$

$\rightarrow$ $f'(x_0) = 0.$

(ii)

$$m \leq f(z) < f(a) = f(b)$$
$$\|$$
$$f(x_1)$$



$$\downarrow$$

$$a < x_1 < b$$

$\hookrightarrow$ 内点 x_1 で極小 $\longrightarrow$ $f'(x_1) = 0$

平均値の定理

$f : [a, b] \longrightarrow \mathbb{R}$ 　(a, b)の各点で微分可能

$[a, b]$ —— 連続

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

を満たす $c \in (a, b)$ が存在する.

$$\varphi(x) := f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$$

$:=$ 左を右で定義.

$$\varphi(a) = f(a) - *\cdot(a-a) = f(a)$$

$$\varphi(b) = f(b) - \frac{f(b)-f(a)}{\cancel{b-a}}\cancel{(b-a)}$$

$$= f(b) - (f(b)-f(a)) = f(a)$$

ある $c \in (a, b)$ において $\varphi'(c) = 0$ by Rolle.

$$\varphi'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

I 開区間　$I = (a, b)$, $(a, +\infty)$, $(-\infty, b)$, $(-\infty, +\infty)$

$f : I \longrightarrow \mathbb{R}$　各点で微分可能

(i)　$f'(x) > 0 \quad (x \in I)$

$x, y \in I,\ \underline{x < y} \implies f(x) < f(y)$

狭義の単調増加.

(ii)　$f'(x) < 0 \quad (x \in I)$

$x, y \in I,\ x < y \implies f(x) > f(y)$

狭義の単調減少

(i) $\dfrac{f(y)-f(x)}{\underset{\substack{\vee \\ 0}}{y-x}} = \underset{\substack{\vee \\ 0}}{f'(c)}$ を満たす $x<c<y$ が存在する.

$\rightarrow \quad f(y)-f(x)>0$

$\rightarrow \quad f(y)>f(x)$

1.3.4

$$f(x) = \frac{x^2}{1+x^2} \qquad f'(x) = \frac{2x}{(1+x^2)^2}$$

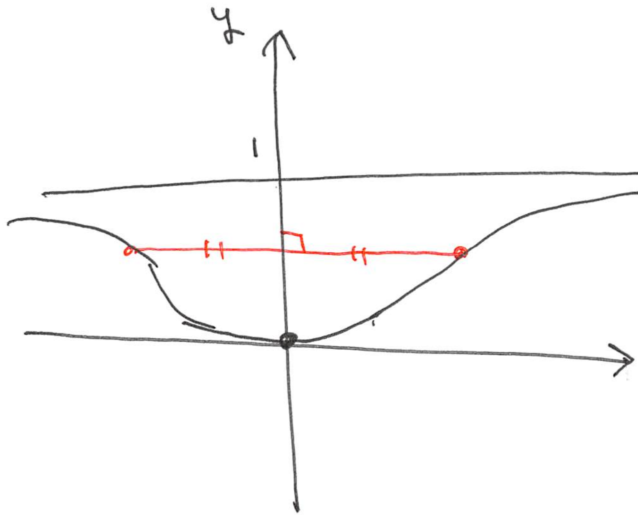
$$f'(x) \gtreqless 0 \iff x \gtreqless 0$$

$$\uparrow$$

$$(1+x^2)^2 > 0$$

x		0	
y'	$-$	0	$+$
y	$\searrow$	0	$\nearrow$

$$f(x) \geqq 0 \iff \begin{cases} x \neq 0 \\ x = 0 \end{cases}$$



$$\frac{x^2}{1+x^2} = \frac{1}{\frac{1}{x^2}+1} \longrightarrow \frac{1}{0+1} = 1$$

$x \to +\infty$ のとき $\frac{1}{x^2} \to 0$

$-\infty$

111 p.

$f(t) = t \log t \quad (t > 0)$

$(fg)' = f'g + fg'$

$f'(t) = (t)' \log t + t (\log t)'$

$(\log t)' = \frac{1}{t}$

$= 1 \cdot \log t + t \cdot \frac{1}{t}$

$= \log t + 1$

$f'(t) \gtreqless 0$

$\iff \log t \gtreqless -1$

$\iff t \gtreqless \frac{1}{e}$

t	(0)		$\frac{1}{e}$	
f'		$-$	0	$+$
f	(0)	$\searrow$	$-\frac{1}{e}$	$\nearrow$

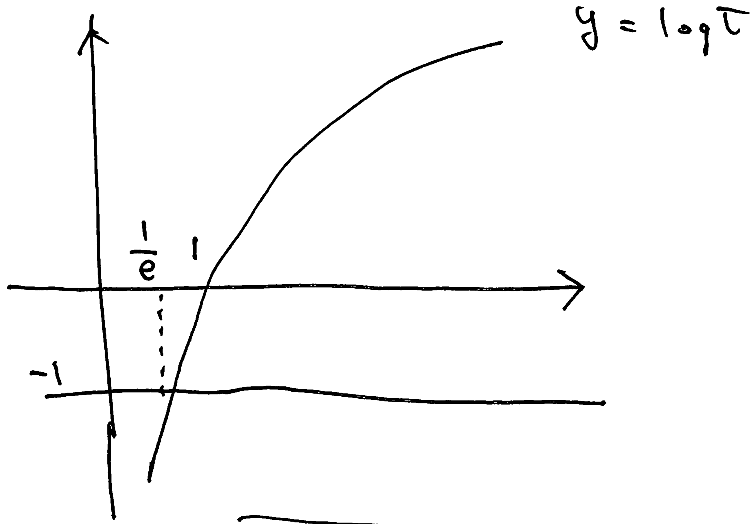
$\frac{1}{e} \cdot \log \frac{1}{e}$

$= -\frac{1}{e}$

$t \to +0$ のとき.

$t \log t \to 0$

$\dfrac{t}{e^t} \to 0 \quad (t \to +\infty)$



$t \to +0$ のとき $\log t \to -\infty$

$s = -\log t \longrightarrow +\infty$

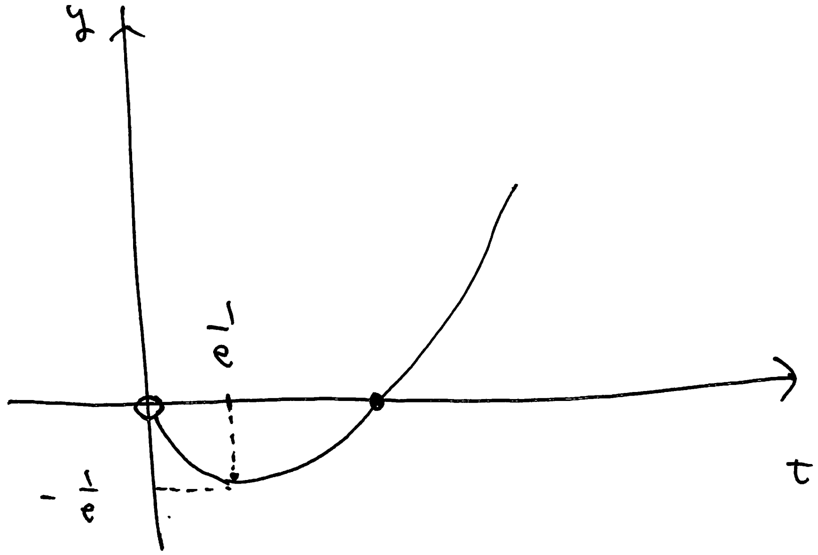
$e^s = -t$

$t = e^{-s}$

$t \log t = e^{-s}(-s) = -\dfrac{s}{e^s} \longrightarrow -0 = 0.$

$0 < x < y$

$\iff \log x < \log y.$



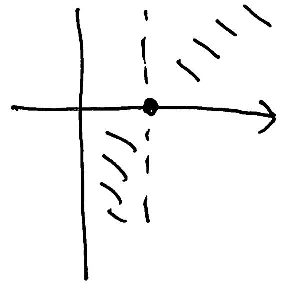
$$y = t\log t \gtreqless 0$$

$$\Updownarrow$$

$$\log t \gtreqless 0$$

$$\Updownarrow$$

$$t \gtreqless 1$$



$$f''(t) = \frac{1}{t} > 0 \leadsto$$ 下に凸

増減表を求める

(1) $\dfrac{x}{1+x^2}$　　(2) xe^{x}　　(3) $x^2\log x$

(4) $\dfrac{\log x}{x}$.

I　$f'(x)$ を求める

(1) $f(x) = (x^2 - 1)^5$　　(2) $f(x) = \dfrac{x}{(1+x^2)^4}$

(3) $f(x) = \left(\dfrac{x+2}{x-1}\right)^3$　　(4) $f(x) = x^2 e^{-x}$.

(5) $f(x) = \dfrac{\log x}{x^2}$　　(6) $f(x) = \log(2x+1)$