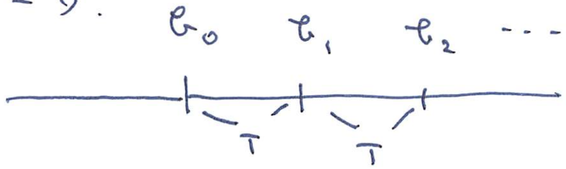
2016/08/05
1st Rec.

6回目で 1変数の微分・積分.

数列. $\longleftrightarrow$ 時系列データ.



初項 b_0

国債. 2%. 年利.

コメント 応用上は 初項を b_0 とする方が 考えやすい(高校では b_1 であった)

0年度 500単位. $b_0=500$.

残高.

毎年 T単位返済.

$$b_1=(500-T)\times 1.02. = (b_0-T)\times 1.02.$$

$$b_2=(b_1-T)\times 1.02.$$

$$\vdots$$

$$b_{n+1}=(b_n-T)\times 1.02$$ ← 差分方程式.

漸化式.

$$\vdots$$

解くのは.まだ.

等差数列 $a_{n+1} - a_n = d$. d 公差.

$$a_0 \xrightarrow{+d} a_1 \xrightarrow{+d} a_2 \longrightarrow a_3 \cdots \xrightarrow{+d} a_n$$

初项 (a_0)

$$\boxed{a_n = a_0 + nd}$$

等比数列 $a_{n+1} = \alpha a_n$

$$a_0 \xrightarrow{\times \alpha} a_1 \xrightarrow{\times \alpha} a_2 \longrightarrow \cdots \xrightarrow{\times \alpha} a_n$$

$$a_n = a_0 \times \alpha^n$$

α 公比.

CT 7p.

差分方程式.

$$\begin{cases} a_{n+1} = 3a_n + 4 & (n = 0, 1, 2, \cdots) \\ a_0 = c \end{cases}$$

初期条件.

$$a_0 \longrightarrow a_1 \longrightarrow a_2 \rightarrow \cdots$$

$$a_1 = 3a_0 + 4, \quad a_2 = 3a_1 + 4$$

$\boxed{\Lambda \quad \lambda \quad \text{ラムダ} \quad L \quad \ell}$

$$\lambda = 3\lambda + 4. \iff \lambda = -2.$$

初期条件を忘れて. $a_n = \lambda \quad (n = 0, 1, 2, 3, \cdots)$

一般の a_n

$$\begin{aligned} a_{n+1} &= 3a_n + 4 \\ -)\quad -2 &= 3\times(-2) + 4 \\ \hline a_{n+1} + 2 &= 3(a_n + 2) \end{aligned}$$

$b_n = a_n + 2$ とすると $b_{n+1} = 3b_n.$ $\rightsquigarrow$

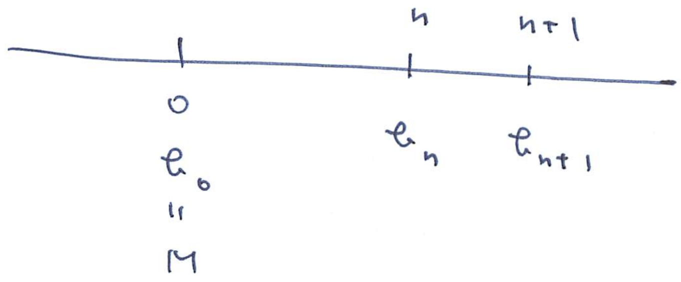
$$b_0 \xrightarrow{\times 3} b_1 \xrightarrow{\times 3} b_2 \xrightarrow{\times 3} \cdots \xrightarrow{\times 3} b_n$$

$$b_n = b_0 \cdot 3^n$$

$$\rightarrow a_n + 2 = (a_0 + 2)\cdot 3^n \longrightarrow \boxed{a_n = (a_0 + 2)\cdot 3^n - 2}$$

8p. 演習1.2.

9p. 借金 M. 現時点



毎年 T 返済.

利率 r.

b_n 年度初めの残高. $\longrightarrow$ $b_1 = (M - T) \times (1 + r)$

$\longrightarrow$ $\boxed{b_{n+1} = (b_n - T) \times (1 + r)}$

$$\lambda = (\lambda - T) \times (1 + r)$$

$\longrightarrow$ $\lambda = \dfrac{T(1+r)}{r} = \lambda_0$ とおく.

b_0

$$
\begin{array}{rl}
 & b_{n+1} = (b_n - T) \times (1 + r) \\
- \,) & \lambda_0 = (\lambda_0 - T) \times (1 + r) \\
\hline
 & b_{n+1} - \lambda_0 = (b_n - \lambda_0) \times (1 + r)
\end{array}
$$

$\{e_n - \lambda_0\}$ は公比 $1+r$ の等比数列

$$e_n - \lambda_0 = (e_0 - \lambda_0)(1+r)^n$$

$$\boxed{e_n = \left(M - \frac{T(1+r)}{r}\right)(1+r)^n + \frac{T(1+r)}{r}}$$

$r > 0$

$M - \dfrac{T(1+r)}{r} > 0$ のとき e_n は増大.

$<$ 減少.

和の公式.

Σ シグマ.

Σ の ⇔ S の.

sum の.

$\int$

$$\sum_{n=0}^{10} a_n = a_0 + a_1 + \cdots + a_{10}.$$

公式

Ⅰ $$\sum_{k=1}^{N} (a_k + b_k) = \sum_{k=1}^{N} a_k + \sum_{k=1}^{N} b_k.$$

$$(a_0 + b_0) + (a_1 + b_1) + \cdots + (a_N + b_N)$$

$$= (a_0 + a_1 + \cdots + a_N) + (b_0 + b_1 + \cdots + b_N)$$

Ⅱ $$\sum_{k=0}^{N} (\lambda b_k) = \lambda \left(\sum_{k=0}^{N} b_k \right)$$

λ：定数

$$(\lambda b_0 + \lambda b_1 + \cdots + \lambda b_N) = \lambda (b_0 + b_1 + \cdots + b_N)$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\begin{array}{ccccc} 1 & 2 & \overset{+1}{\curvearrowright} 3 & \cdots & n \\ n & n-1 & n-2 & \cdots & 1 \\ & \underset{-1}{\curvearrowright} & & & \\ n+1 & n+1 & n+1 & \cdots & n+1 \end{array}$$

$$= \sum_{k=0}^{n} k$$

等差数列 $a_n = a_0 + nd$. 公差 d.

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} a_0 + d\sum_{k=0}^{n} k = a_0(n+1) + d\,\frac{n(n+1)}{2}$$

$$= \sum_{k=0}^{n} (a_0 + kd)$$

1

$$= \frac{(2a_0 + nd)}{2}(n+1)$$

$$= \frac{a_0 + \underbrace{(a_0 + nd)}_{= a_n}}{2}(n+1).$$

$$a_0 \xrightarrow{+d} a_1 \xrightarrow{+d} a_2 \longrightarrow \cdots \xrightarrow{+d} a_n$$

$$a_n \xleftarrow[-d]{+d} a_{n-1} \qquad \xleftarrow[-d]{+d} a_0$$

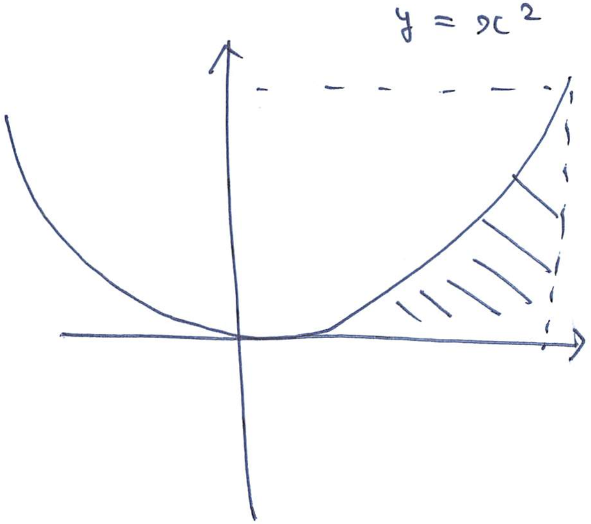
等比数列.　公比 r　$r \neq 1$　$r=1$ の場合は定数列

$$\underbrace{a_0,\ a_0,\ \cdots,\ \overset{a_n}{\overset{\shortparallel}{a_0}}}_{(n+1)a_0}$$

$$
\begin{array}{rrl}
 & S_n = & a_0 + a_0 r + \cdots + a_0 r^n \\
- \big) & r S_n = & \phantom{a_0 + {}} a_0 r + \cdots + a_0 r^n + a_0 r^{n+1} \\
\hline
 & (1-r) S_n = & a_0 \qquad\qquad\qquad - a_0 r^{n+1}
\end{array}
$$

$$\longrightarrow \quad S_n = \frac{a_0(1-r^{n+1})}{1-r}$$

$r \neq 1$



$$S = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\left(\frac{x^3}{3}\right)' = x^2$$

準備.

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \sum_{k=1}^{n} (k^2+k) - \sum_{k=1}^{n} k$$

$$k^2 + k = k(k+1)$$

$$k^2 = (k^2+k) - k$$

$$= \frac{1}{3}\left\{k(k+1)(k+2) - (k-1)k(k+1)\right\}$$

$k=1 \qquad \frac{1}{3}\left(\cancel{1\cdot 2\cdot 3} - 0\cdot 1\cdot 2\right)$

$k=2 \qquad \frac{1}{3}\left(\cancel{2\cdot 3\cdot 4} - \cancel{1\cdot 2\cdot 3}\right)$

$k=3 \qquad \frac{1}{3}\left(3\cdot 4\cdot 5 - \cancel{2\cdot 3\cdot 4}\right)$

$\vdots$

$k=n \qquad \frac{1}{3}\left(n(n+1)(n+2) - \cancel{(n-1)n(n+1)}\right\}$

$+)$

$$\sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2)$$

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n}(k^2+k) - \sum_{k=1}^{n} k$$

$$= \frac{1}{3}n(n+1)(n+2) - \frac{1}{2}n(n+1)$$

$$= n(n+1)\left(\frac{1}{3}(n+2) - \frac{1}{2}\right) = \frac{2n+1}{6}\,n(n+1)$$

公式 $\quad \displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$$k^3 = k^3 - k = (k-1)k(k+1)$$

$$= \frac{1}{4}\{(k-1)k(k+1)(k+2) - (k-2)(k-1)k(k+1)\}.$$

ここで $\displaystyle\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n}(k^3 - k) + \sum_{k=1}^{n} k$

$$\sum_{k=1}^{n}(k^3 - k) = \frac{1}{4}\{n(n+1)(n+2)(n+3) - (1-2)(1-1)1\cdot(1+1)\}$$

$$= \frac{1}{4}n(n+1)(n+2)(n+3)$$

より

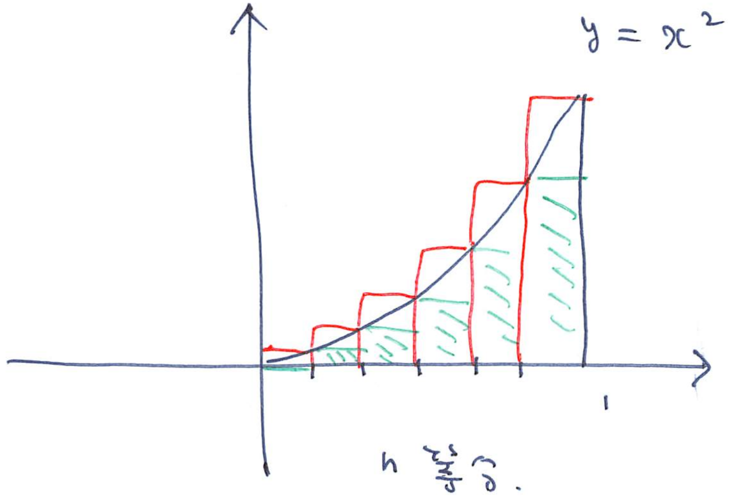
$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n(n+1)(n+2)(n+3) + \frac{1}{2}n(n+1)$$

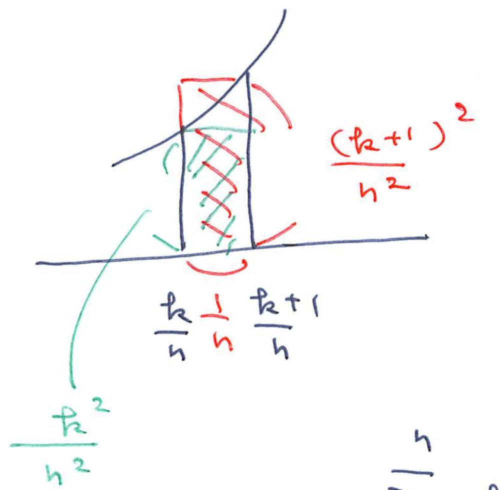
$$= \frac{1}{4}n(n+1)\{(n+2)(n+3) + 2\}$$

$$= \frac{1}{4}n(n+1)(n^2 + 5n + 8)$$

積分 = 面積を求めている.

152p.





$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

外側の面積の和.

$$= \frac{1}{n}\left(\frac{1}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{n^2}{n^2}\right)$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

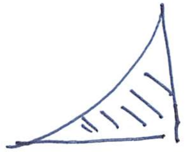
$$= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \longrightarrow \frac{1}{3} \quad (n \to +\infty)$$

内側の面積. $= \frac{1}{n}\left(\frac{0^2}{n^2} + \frac{1^2}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{(n-1)^2}{n^2}\right)$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + (n-1)^2\right) = \frac{1}{n^3} \cdot \frac{(n-1)n(2n-1)}{6} = \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$\longrightarrow \frac{1}{3} \quad (n \to +\infty)$$

はさみうちの原理

$\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$ <  < $\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$\downarrow$ $\downarrow$ $\downarrow$

$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$

 nに依らない  $= \frac{1}{3}$

---

1) $a_n \leq b_n \leq c_n$ $a_n \to \alpha,\ c_n \to \alpha \quad (n \to +\infty)$ CT 23p.

$\Longrightarrow b_n \to \alpha.$

2) $a_n = \alpha$ 一定 $\Longrightarrow a_n \to \alpha$

1) これは第1回目の最後に示している。

2項定理.

$$(x+y)^2 = x^2 + 2xy + y^2$$

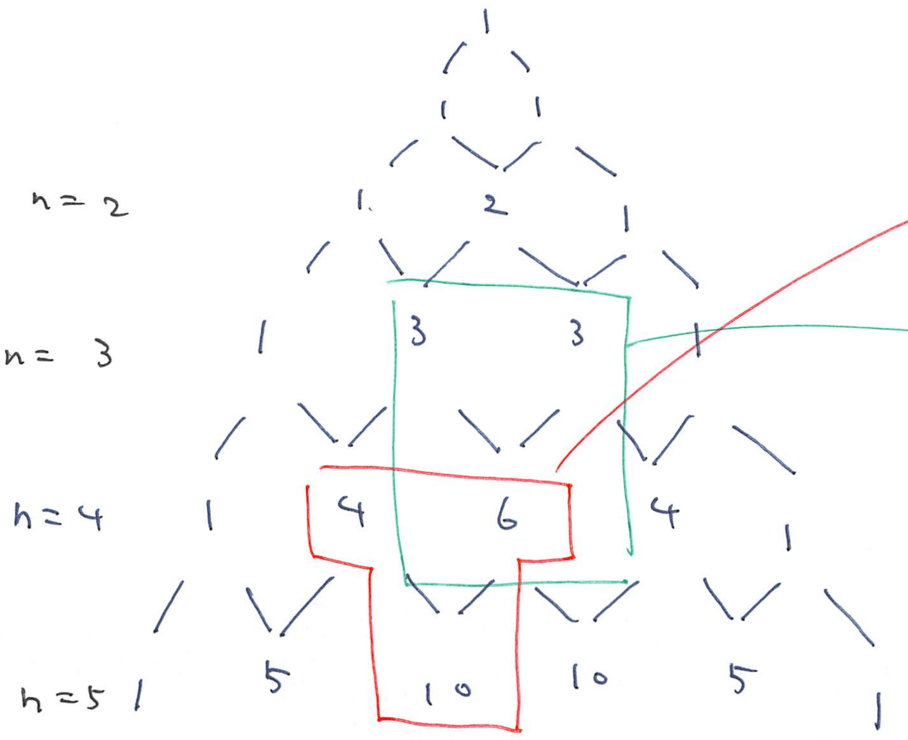
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = \boxed{x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4}$$

$$\begin{array}{ccccc} & 1 & 3 & 3 & 1 \\ 1 & 3 & 3 & 1 & \\ \hline 1 & 4 & 6 & 4 & 1 \end{array}$$

$$x^3y + 3x^2y^2 + 3xy^3 + y^4$$

$$x^4 + 3x^3y + 3x^2y^2 + 3xy^3$$



$${}_4C_1 + {}_4C_2 = {}_5C_2$$

$${}_3C_1 + {}_3C_2 = {}_4C_2$$

Pascal の三角形.

$$(x+y)^n = \sum_{k=0}^{n} {}_nC_k \, x^k y^{n-k}$$

説明.

$$\overbrace{(x+y)(x+y)\cdots(x+y)}^{n\text{個}}$$

2^n 個の項が出てくる.

$x^k y^{n-k}$ $(k=0,\cdots,n)$

$x y^{n-1}$ $\quad {}_nC_1 = n.$

$x^2 y^{n-2}$ $\quad {}_nC_2 = \dfrac{n(n-1)}{2}.$

$\vdots$

$x^{n-1} y$ $\quad {}_nC_{n-1} = {}_nC_1 = n.$

$x^k y^{n-k}$ $\quad {}_nC_k$ 個.

$$= \frac{n!}{k!(n-k)!}$$

(#) $\boxed{{}_{n-1}C_{k-1} + {}_{n-1}C_{k} = {}_{n}C_{k}}$

演習

$${}_{n}C_{k} = \frac{n!}{k!(n-k)!}$$

を用いて(#)を示せ.

①　②　---　ⓝ

(i)　ⓝ をとる場合.　①　--　(n−1) から $k-1$ 個とる.　${}_{n-1}C_{k-1}$

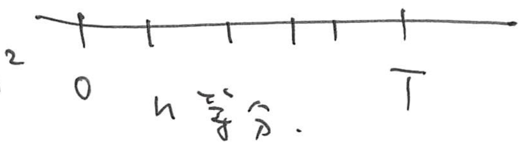
(ii)　——　とらない　——　k 個とる　${}_{n-1}C_{k}$.

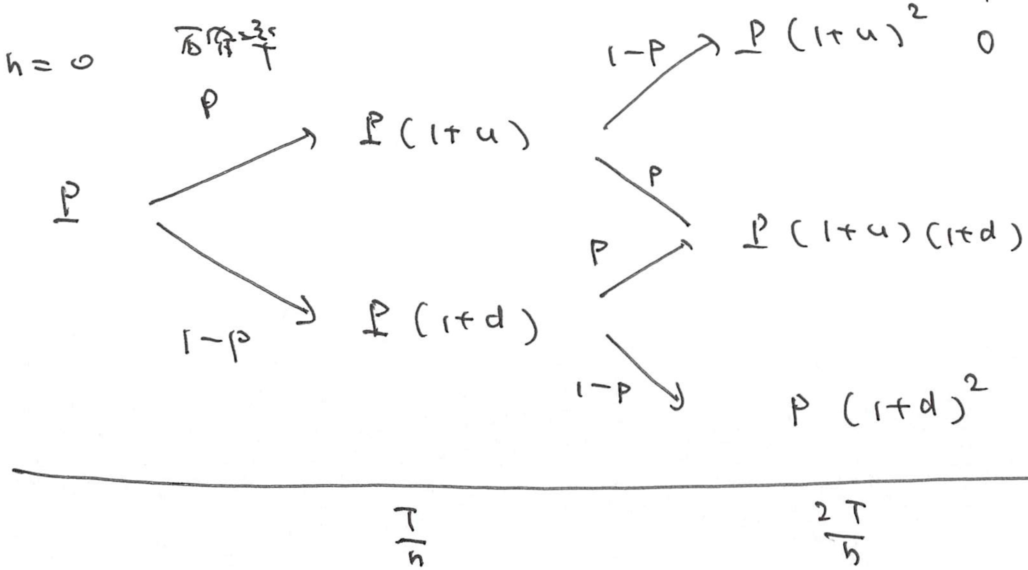
2項モデル

A社の株価

$u > 0 > d$.

$u > d$.





$n \to +\infty$

とすると

ブラック・ショールズの公式を導くことができる.

数列の収束. 18p.

$a_0, a_1, a_2, \cdots$

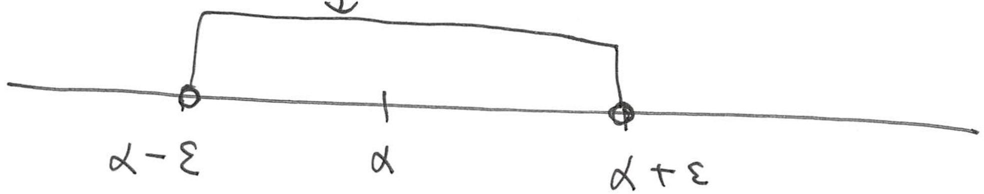
$a_n \longrightarrow \alpha \quad (n \longrightarrow +\infty)$

$\iff$ $\forall \varepsilon > 0 \ \exists N$ 番号

定義

$\left(n \geqq N \Longrightarrow \alpha - \varepsilon < a_n < \alpha + \varepsilon \right)$

$a_N, a_{N+1}, a_{N+2}, \cdots$



$a_0, a_1, \cdots, a_{N-1}$ は分からない

2項定理は16p〜

$\mathbb{R}$ 実数全体の集合

$\alpha \in \mathbb{R}$　　$\in$ E　e E.

属する　　epsilon

element 要素.

$\forall$ すべての　All

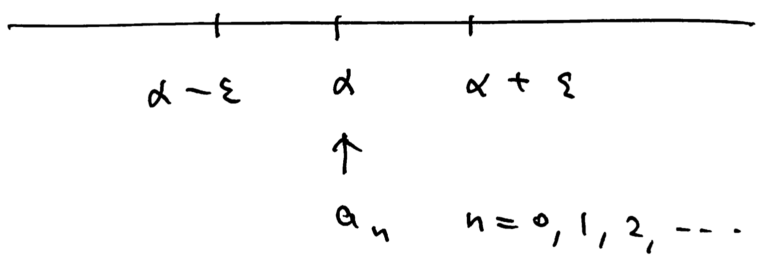
$\exists$ ある　Exist

École Polytechnique.

ナポレオンが設けた学校. 理工科学校と呼ばれるが、そこで微分積分を教える必要から収束の定義が厳密に考えられるようになった.

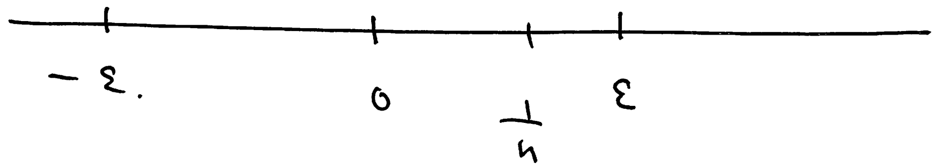
(i) $a_n = \alpha \quad (n = 0, 1, 2, 3, \cdots) \qquad a_n \to \alpha$

$\forall \varepsilon > 0$ が…与えられたとする



(ii) $a_n = \frac{1}{n} \quad (n = 1, 2, 3, \cdots) \qquad \frac{1}{n} \longrightarrow 0 \quad (n \to +\infty)$

$\varepsilon > 0$



$$\frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n.$$

ガウス記号.

$[1.3] = 1$

$[1.0] = 1$

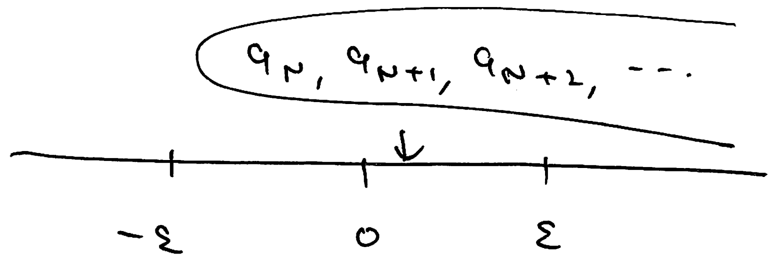
$[2.1] = 2$

$[x] + 1 > x.$

$N = \left[\frac{1}{\varepsilon}\right] + 1.$ とすれば

$$n \geq N > \frac{1}{\varepsilon} \longrightarrow \frac{1}{\varepsilon} < n \longrightarrow \frac{1}{n} < \varepsilon$$

$$n \geq N := \left[\frac{1}{\varepsilon}\right] + 1 \implies -\varepsilon < 0 < \frac{1}{n} < \varepsilon$$



定理 $a_n \to \alpha,\ b_n \to \beta \quad (n \to +\infty)$ (20p)

(i) $a_n \pm b_n \to \alpha \pm \beta \quad (n \to +\infty)$

(ii) $a_n \cdot b_n \to \alpha\beta \quad (n \to +\infty)$

(iii) $a_n \neq 0,\ \alpha \neq 0 \implies \dfrac{b_n}{a_n} \to \dfrac{\beta}{\alpha}$

すべてのn

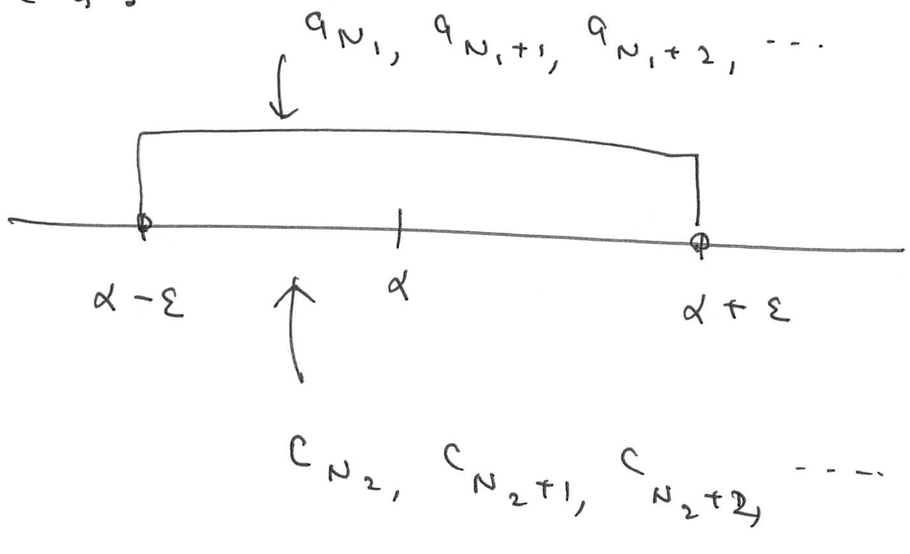
この定理の証明は省略.

定理 $a_n \to \alpha,\ c_n \to \alpha,\quad a_n \leq b_n \leq c_n$ (すべてのn)

$\implies b_n \to \alpha.$ (23p)

はさみうちの定理 The Squeeze Theorem.

$\forall \varepsilon > 0$ が与えられたとする

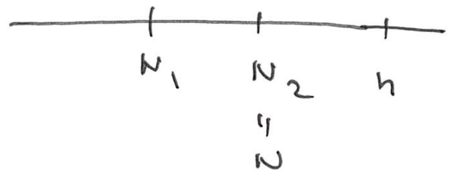


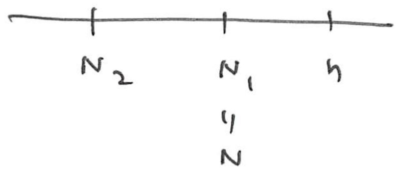
$\exists N_1 \quad \exists N_2$

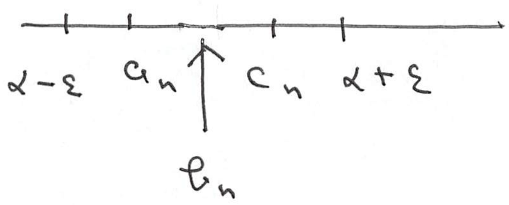
$N = \max(N_1, N_2)$

$n \geq N \longrightarrow n \geq N_1 \longrightarrow \alpha - \varepsilon < a_n < \alpha + \varepsilon$

$ \searrow n \geq N_2. \longrightarrow \alpha - \varepsilon < c_n < \alpha + \varepsilon$







$\longrightarrow \quad \alpha - \varepsilon < a_n < c_n < \alpha + \varepsilon.$

$\longrightarrow \quad \alpha - \varepsilon < a_n < b_n < c_n < \alpha + \varepsilon.$

$$\left(\begin{array}{l} n \geqq N \Longrightarrow \\ \qquad \alpha - \varepsilon < b_n < \alpha + \varepsilon \end{array} \right)$$

$b_n \to \alpha$ がわかる.

§3(1) 1.2. 22p.

(i) $\frac{1}{n^2} = \frac{1}{n}\cdot\frac{1}{n} \longrightarrow 0\cdot 0 = 0.$

$\frac{1}{n^3} = \frac{1}{n^2}\cdot\frac{1}{n} \longrightarrow 0\cdot 0 = 0$

$k = 1, 2, 3, \dots$ $\quad \frac{1}{n^k} \longrightarrow 0 \quad (n \to +\infty)$

$\frac{3}{n} = 3\cdot\frac{1}{n} \longrightarrow 3\cdot 0 = 0$

(ii) $a_n = \frac{1}{3+n} = \frac{1}{n}\cdot\frac{1}{\frac{3}{n}+1}$

$\frac{3}{n}+1 \longrightarrow 0+1 = 1$

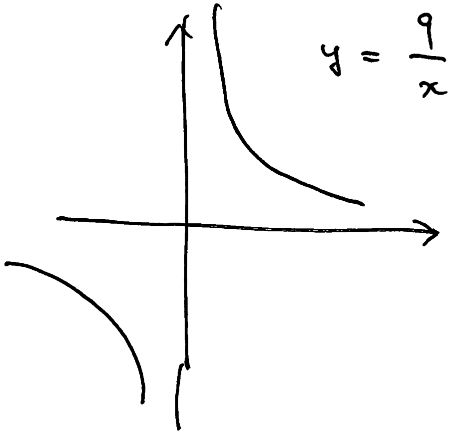
$\frac{1}{\frac{3}{n}+1} \longrightarrow \frac{1}{1} = 1$

$\longrightarrow 0\cdot 1 = 0.$

(iii) $a_n = \frac{4-n}{5+n} = \frac{\frac{4}{n}-1}{\frac{5}{n}+1} \longrightarrow \frac{4\cdot 0-1}{5\cdot 0+1} = -1.$

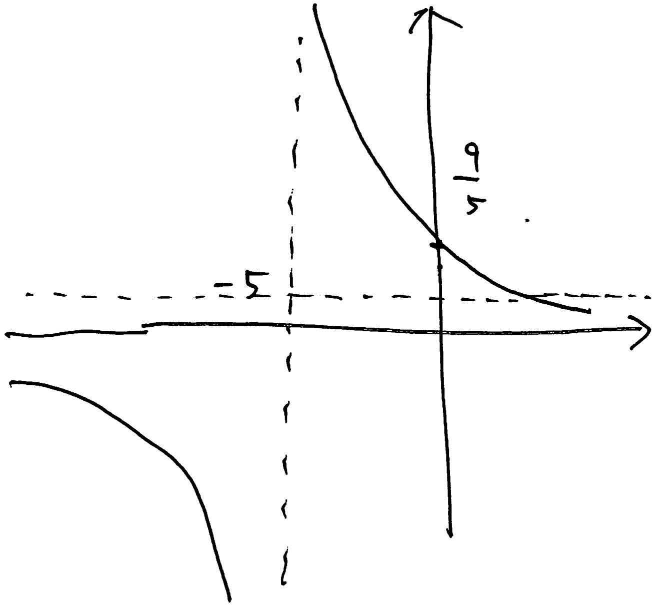
$$y = \frac{4-x}{5+x} = \frac{-x+4}{x+5} = \frac{-(x+5)+9}{x+5} = -1 + \frac{9}{x+5}$$

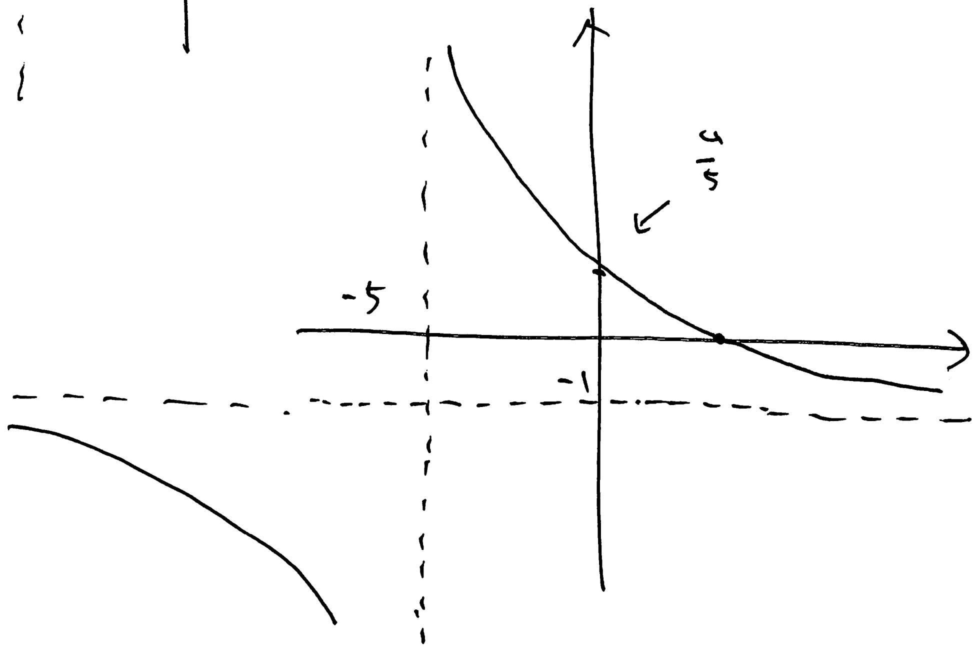
$$y = \frac{9}{x}$$



$$y = \frac{9}{x+5}$$

$$\frac{-x+4}{x-5}$$





演習

$$y = \frac{x-2}{x+1}$$

$$y = \frac{2x-1}{x-1}$$

$$\boxed{|r|<1 \text{ のとき } \quad r^n \longrightarrow 0 \quad (n \longrightarrow +\infty)}$$

24p.

$0<r<1$ のとき. $s = \dfrac{1}{r}$ とする　$s>1$ となる.

$s = 1+\theta$ とすると $\theta > 0$.

$$s^n = (1+\theta)^n$$

$$= 1 + {}_nC_1 \theta + {}_nC_2 \theta^2 + \cdots + {}_nC_k \theta^k + \cdots + \theta^n$$

${}_nC_1 = n$

${}_nC_k \theta^k > 0$

$$> n\theta.$$

これをおとすと

$n r^n \to 0$ もいえる.

$$0 < r^n = \frac{1}{s^n} < \frac{1}{n\theta}$$

$0 \to 0$, $r^n \to 0$, $\dfrac{1}{n\theta} \to 0 \cdot \dfrac{1}{\theta} = 0$

はさみうち.

$$S_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n \longrightarrow S \quad (n \to +\infty)$$

のとき.

$$\sum_{k=0}^{+\infty} a_k = S \quad \text{と書く.}$$

$|r| < 1$

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1-r} = \frac{1}{1-r} - \frac{r}{1-r} \cdot r^n$$

$$\longrightarrow \frac{1}{1-r} - \frac{r}{1-r} \cdot 0 = \frac{1}{1-r}$$

乗数効果，信用創造 (31p, 32p).

演習 1.9.* $|r| < 1$ のとき $n r^n \longrightarrow 0 \quad (n \to +\infty)$

ヒント. (1.34).

44p〜

重点 80p〜90p.