$z = 5x^2 + 4xy + 2y^2 - 2x + 4y.$

$\equiv 5(x-a)^2 + 4(x-a)(y-b) + 2(y-b)^2 + c$

$\uparrow$

両辺を x で偏微分する.
y

$$\begin{cases} 10x + 4y - 2 = 10(x-a) + 4(y-b) \\ 4x + 4y + 4 = 4(x-a) + 4(y-b) \end{cases}$$

$$\rightsquigarrow \begin{cases} 10a + 4b = 2 \\ 4a + 4b = -4 \end{cases}$$

$$a = \frac{\begin{vmatrix} 2 & 4 \\ -4 & 4 \end{vmatrix}}{\begin{vmatrix} 10 & 4 \\ 4 & 4 \end{vmatrix}} = \frac{24}{24} = 1$$

$$b = \frac{\begin{vmatrix} 10 & 2 \\ 4 & -4 \end{vmatrix}}{24} = \frac{-48}{24} = -2.$$

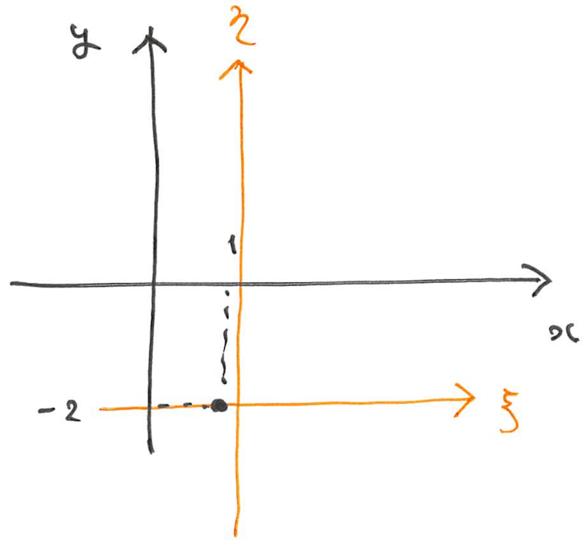
$5 - 8 + 8 - 2 - 8$
$= -5$

$\underbrace{x = a}_{=1},\ y = b = -2$ を代入.

$-5 = c$

$$z = 5(x-1)^2 + 4(x-1)(y+2) + 2(y+2)^2 + 5$$

$$\begin{cases} \xi = x - 1 \\ \eta = y + 2 \end{cases}$$



$$= 5\xi^2 + 4\xi\eta + 2\eta^2 + 5$$

$$= \left(\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right) + 5$$

$$5 > 2$$

$$\begin{vmatrix} 5 & 2 \\ 2 & 2 \end{vmatrix} = 10 - 4 = 6 > 0$$

$$> 0 \quad \Leftarrow \quad \begin{pmatrix} \xi \\ \eta \end{pmatrix} \neq \vec{0}$$

$$\Updownarrow$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$> 5$$

$$\uparrow$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$z = \left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) + \left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \qquad \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} x - a \\ y - b \end{pmatrix}$$

$${}^{t}A = A$$

$$= \left(A\left(\begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}\right), \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}\right)$$

$$+ \left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}\right)$$

$$= \left(A\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) + \left(A\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right) + \left(A\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) + \left(A\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right)$$

$$\left(A\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right) = \left(\begin{pmatrix} \xi \\ \eta \end{pmatrix}, {}^{t}A\begin{pmatrix} a \\ b \end{pmatrix}\right) = \left(\begin{pmatrix} \xi \\ \eta \end{pmatrix}, A\begin{pmatrix} a \\ b \end{pmatrix}\right)$$

$$+ \left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) + \left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right)$$

$$= \left(A \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right) + \left(\underbrace{2A \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix}}_{= \vec{0}}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right)$$

$$+ \underbrace{\boxed{\left(A \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right) + \left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right)}}_{= f(x = a,\ y = b)}$$

$$2A \begin{pmatrix} a \\ b \end{pmatrix} = - \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$|A| = 10 - 4 = 6 \neq 0$$

各自.

$$\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{2} A^{-1} \begin{pmatrix} -2 \\ 4 \end{pmatrix} = -\frac{1}{2} \cdot \frac{1}{6} \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \cdots = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$= \left(-\frac{1}{2} \begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) + \left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right)$$

$$= \frac{1}{2} \left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right)$$

$$z = (A\vec{x}, \vec{x}) + (\vec{b}, \vec{x}) + c$$

A : $n \times n$ 対称

正定値 $\rightsquigarrow |A| > 0$

$$\vec{\xi} = \vec{x} - \vec{\alpha}$$

$$= (A\vec{\xi}, \vec{\xi}) + c'$$

$$\vec{\alpha} = -\frac{1}{2}A^{-1}\vec{b} \iff A\vec{\alpha} = -\frac{1}{2}\vec{b}$$

$\vec{x} = \vec{\alpha}$ とおくと $\vec{\xi} = \vec{0}$

$$\boxed{(A\vec{\alpha}, \vec{\alpha}) + (\vec{b}, \vec{\alpha}) + c = c'}$$

$$\parallel$$

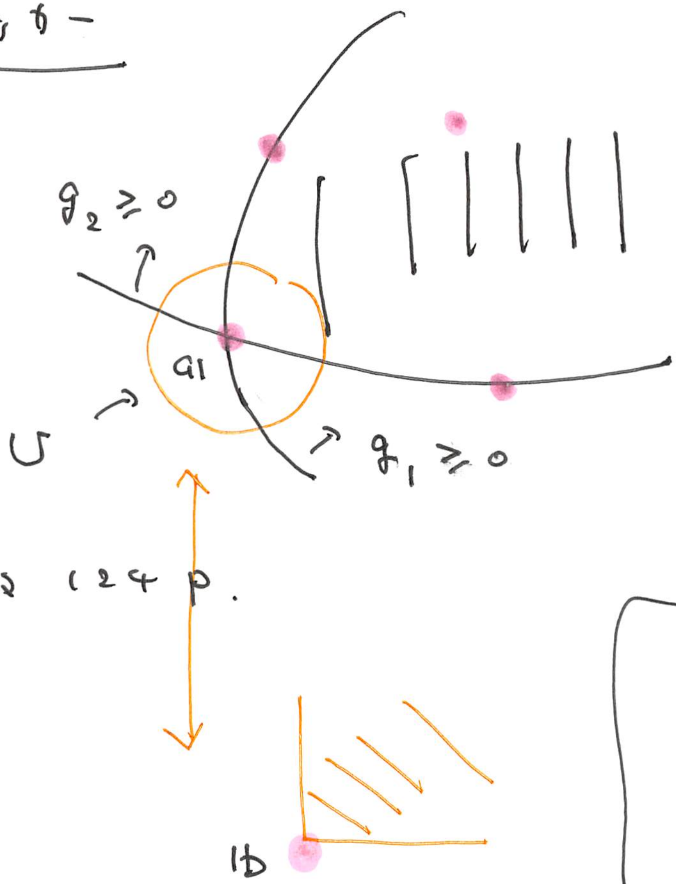
$$\left(-\frac{1}{2}\vec{b}, \vec{\alpha}\right) + (\vec{b}, \vec{\alpha}) + c = \frac{1}{2}(\vec{b}, \vec{\alpha}) + c$$

講義中
ここ間違って
います.
修正済

$\vec{x} = -\frac{1}{2}A^{-1}\vec{b}$ で 最小値 $\frac{1}{2}(\vec{b}, \vec{\alpha}) + c$ をとる

2次計画法.

クーン・タッカー



参考 124 p.

$$\{(x, y);\ g_1(x, y) \geqq 0,\ g_2(x, y) \geqq 0\}$$

$\in (x, y)$ の下で

$z = f(x, y)$

$$G : U \longrightarrow \mathbb{R}^2 \ni (\xi, \eta)$$

$$(x, y) \longmapsto (g_1(x, y),\ g_2(x, y))$$

$$b = (g_1(a),\ g_2(a))$$

$$D(G) = \begin{pmatrix} g_{1x} & g_{1y} \\ g_{2x} & g_{2y} \end{pmatrix}$$

関数行列

Jacobi行列.

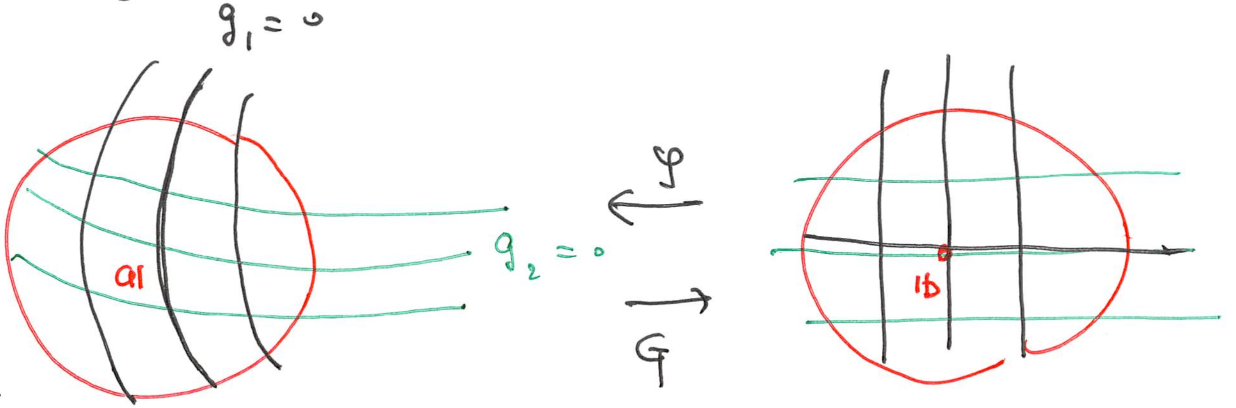
前提 $|D(G)|(a) \neq 0$.

$\longrightarrow$ 逆関数定理

$U \supset U_0$ 開, $U_0 \ni \mathbf{a}$

V_0 開, $V_0 \ni \mathbf{b}$

$U_0 \overset{1:1}{\longleftrightarrow} V_0$, $U_0 \ni \mathbf{a}$, $V_0 \ni \mathbf{b}$



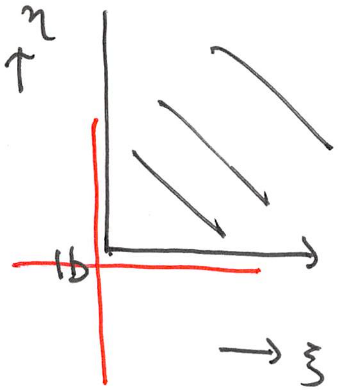
正確には

$\varphi : V_0 \longrightarrow U_0$ 逆写像.

$G \circ \varphi(\xi, \eta) = (\xi, \eta)$

$\varphi \circ G(x, y) = (x, y)$

$$F(\xi, \eta) = f(\varphi_1(\xi,\eta), \varphi_2(\xi,\eta))$$



f が $\mathbb{a}$ で極大 $\Longleftrightarrow$ F が $\mathbb{b}$ で極大.

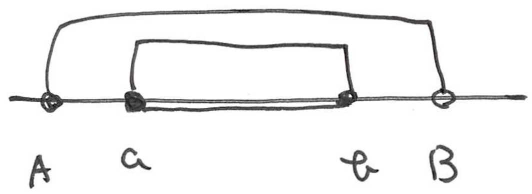
$\longrightarrow$

$F(\xi, b_2)$, $F(b_1, \eta)$

$F_\xi(\mathbb{b}) \leq 0$

$F_\eta(\mathbb{b}) \leq 0$

$f : (A, B) \longrightarrow \mathbb{R}$



a で f が $[a, b]$ で 極大

$\Longrightarrow f'(a) \leq 0$

$$\left\{\begin{aligned} \frac{\partial F}{\partial \xi} &= \frac{\partial f}{\partial x}\cdot\frac{\partial \varphi_1}{\partial \xi} + \frac{\partial f}{\partial y}\cdot\frac{\partial \varphi_2}{\partial \xi} \\ \frac{\partial F}{\partial \eta} &= \frac{\partial f}{\partial x}\cdot\frac{\partial \varphi_1}{\partial \eta} + \frac{\partial f}{\partial y}\cdot\frac{\partial \varphi_2}{\partial \eta}. \end{aligned}\right.$$

$$x = \varphi_1(\xi, \eta)$$
$$y = \varphi_2(\xi, \eta)$$

$$f(\varphi_1(\xi,\eta), \varphi_2(\xi,\eta))$$

$$\downarrow$$

$$\begin{pmatrix} \frac{\partial F}{\partial \xi} & \frac{\partial F}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi_1}{\partial \xi} & \frac{\partial \varphi_1}{\partial \eta} \\ \frac{\partial \varphi_2}{\partial \xi} & \frac{\partial \varphi_2}{\partial \eta} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \frac{\partial F}{\partial \xi}(\mathbf{b}) & \frac{\partial F}{\partial \eta}(\mathbf{b}) \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(\mathbf{a}) & \frac{\partial f}{\partial y}(\mathbf{a}) \end{pmatrix} \boxed{\begin{pmatrix} \frac{\partial \varphi_1}{\partial \xi}(\mathbf{b}) & \frac{\partial \varphi_1}{\partial \eta}(\mathbf{b}) \\ \frac{\partial \varphi_2}{\partial \xi}(\mathbf{b}) & \frac{\partial \varphi_2}{\partial \eta}(\mathbf{b}) \end{pmatrix}}$$

$$\shortparallel$$

$$D(G)(\mathbf{a})^{-1}$$

$$G \circ \varphi(\xi, \eta) = (\xi, \eta)$$

$$g_1(\varphi_1(\xi,\eta), \varphi_2(\xi,\eta)) = \xi$$

$$g_2(\varphi_1(\xi,\eta), \varphi_2(\xi,\eta)) = \eta$$

$\xrightarrow[\text{chain rule}]{}$ $\longrightarrow$ $\searrow$ $D(G)(\varphi(\xi,\eta)) \cdot D(\varphi)(\xi,\eta) = I_2$

$$\longrightarrow \quad \underset{(0\ \ 0)}{\underset{\rotatebox{90}{$\neq$}}{(\lambda_1\ \ \lambda_2)}} = -\left(f_x(a)\ \ f_y(a)\right)\begin{pmatrix} g_{1x}(a) & g_{1y}(a) \\ g_{2x}(a) & g_{2y}(a) \end{pmatrix}^{-1}$$

<u>定理</u>

$$L = f + \lambda_1 g_1 + \lambda_2 g_2$$

$x = a$ で 極大

$$L_{x_1}(a) = L_{x_2}(a) = 0 \text{ を満たす } \lambda_1, \lambda_2 \geq 0.$$

$$A = \begin{pmatrix} 3 & -1 & -2 \\ -1 & 3 & 2 \\ -2 & 2 & 6 \end{pmatrix}$$ が正定値の2次形式を定めることを示せ.

解答

A：3次対称行列において

$$\left(A\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) > 0 \quad \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq \vec{0}\right)$$

$\Longleftrightarrow \quad a_{11} > 0, \ |A_2| > 0, \ |A| > 0$

$\Longleftrightarrow \quad A$ の固有値を α, β, γ とすると $\alpha, \beta, \gamma > 0$

(解1)

$A = (a_{ij})$ とすると

$$a_{11} = 3 > 0$$

$$|A_2| = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 9 - 1 = 8 > 0$$

$$\begin{array}{rrrr} & 3 & -1 & -2 \\ +) & -3 & 9 & 6 \\ \hline & 0 & 8 & 4 \end{array} \qquad \begin{array}{rrrr} & -2 & 2 & 6 \\ -) & -2 & 6 & 4 \\ \hline & 0 & -4 & 2 \end{array}$$

$$|A| = \begin{vmatrix} 3 & -1 & -2 \\ -1 & 3 & 2 \\ -2 & 2 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 8 & 4 \\ -1 & 3 & 2 \\ 0 & -4 & 2 \end{vmatrix} = -(-1)\begin{vmatrix} 8 & 4 \\ -4 & 2 \end{vmatrix} = 32 > 0$$

より A が定める2次形式は正定値であることが分かる.

(解2)

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda-3 & 1 & 2 \\ 1 & \lambda-3 & -2 \\ 2 & -2 & \lambda-6 \end{vmatrix} \overset{1行+2行}{=} \begin{vmatrix} \lambda-2 & \lambda-2 & 0 \\ 1 & \lambda-3 & -2 \\ 2 & -2 & \lambda-6 \end{vmatrix}$$

$$= (\lambda-2) \begin{vmatrix} 1 & 1 & 0 \\ 1 & \lambda-3 & -2 \\ 2 & -2 & \lambda-6 \end{vmatrix} = (\lambda-2) \begin{vmatrix} 1 & 1 & 0 \\ 0 & \lambda-4 & -2 \\ 0 & -4 & \lambda-6 \end{vmatrix}$$

$$= (\lambda-2) \begin{vmatrix} \lambda-4 & -2 \\ -4 & \lambda-6 \end{vmatrix} = (\lambda-2)\{(\lambda-4)(\lambda-6)-8\}$$

$$= (\lambda-2)(\lambda^2-10\lambda+16) = (\lambda-2)^2(\lambda-8)$$

より Aの固有値は $2, 2, 8 > 0$ となる. よって Aが定める2次形式は正定値である.

補足 $\quad G(x, y) = (g_1(x, y), g_2(x, y))$

$G \circ \varphi(\xi, \eta) = (\xi \ \eta)$

すなわち

$$\begin{cases} g_1(\varphi_1(\xi, \eta), \varphi_2(\xi, \eta)) = \xi \\ g_2(\varphi_1(\xi, \eta), \varphi_2(\xi, \eta)) = \eta \end{cases}$$

両辺をξで偏微分する.

$$(1) \quad \begin{cases} g_{1x}(\varphi(\xi, \eta)) \cdot \dfrac{\partial \varphi_1}{\partial \xi} + g_{1y}(\varphi(\xi, \eta)) \cdot \dfrac{\partial \varphi_2}{\partial \xi} = 1 \\ g_{2x}(\varphi(\xi, \eta)) \cdot \dfrac{\partial \varphi_1}{\partial \xi} + g_{2y}(\varphi(\xi, \eta)) \dfrac{\partial \varphi_2}{\partial \xi} = 0 \end{cases}$$

両辺をηで偏微分する

$$(2) \quad \begin{cases} g_{1x}(\varphi(\xi, \eta)) \cdot \dfrac{\partial \varphi_1}{\partial \eta} + g_{1y}(\quad) \dfrac{\partial \varphi_2}{\partial \eta} = 0 \\ g_{2x}(\varphi(\xi, \eta)) \dfrac{\partial \varphi_1}{\partial \eta} + g_{2y}(\quad) \dfrac{\partial \varphi_2}{\partial \eta} = 1 \end{cases}$$

(1) は

$$\begin{pmatrix} g_{1x} & g_{1y} \\ g_{2x} & g_{2y} \end{pmatrix} \begin{pmatrix} \varphi_{1\xi} \\ \varphi_{2\xi} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(2) は

$$\begin{pmatrix} g_{1x} & g_{1y} \\ g_{2x} & g_{2y} \end{pmatrix} \begin{pmatrix} \varphi_{1\eta} \\ \varphi_{2\eta} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

となるので

$$\begin{pmatrix} g_{1x} & g_{1y} \\ g_{2x} & g_{2y} \end{pmatrix} \begin{pmatrix} \varphi_{1\xi} & \varphi_{1\eta} \\ \varphi_{2\xi} & \varphi_{2\eta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

よって

$$D(G)\,D(\varphi) = I_2$$

を得る。よって

$$\mathbb{O} = (F_\xi(\mathbb{b}) \; F_\eta(\mathbb{b}))$$

$$= (f_x(\mathbb{a}) \quad f_y(\mathbb{a})) \begin{pmatrix} \varphi_{1\xi}(\mathbb{b}) & \varphi_{1\eta}(\mathbb{b}) \\ \varphi_{2\xi}(\mathbb{b}) & \varphi_{2\eta}(\mathbb{b}) \end{pmatrix}$$

$$= (f_x(\mathbb{a}) \quad f_y(\mathbb{a}))\; D(G)(\mathbb{a})^{-1}$$

となる。

$$\mathbb{O} \leq (\lambda_1 \quad \lambda_2) = -(f_x(\mathbb{a}) \quad f_y(\mathbb{a}))\; D(G)(\mathbb{a})^{-1}$$

と定める。さらに Lagrange 関数を

$$L = f + \lambda_1 g_1 + \lambda_2 g_2$$

と定める。

$$L_x(\mathbb{a}) = f_x(\mathbb{a}) + \lambda_1 g_{1x}(\mathbb{a}) + \lambda_2 g_{2x}(\mathbb{a})$$

$$L_y(\mathbb{a}) = f_x(\mathbb{a}) + \lambda_1 g_{1y}(\mathbb{a}) + \lambda_2 g_{2y}(\mathbb{a})$$

$$(L_x(\mathbb{a}) \quad L_y(\mathbb{a})) = (f_x(\mathbb{a}) \quad f_y(\mathbb{a}))$$

$$+ (\lambda_1 \quad \lambda_2)\begin{pmatrix} g_{1x}(\mathbb{a}) & g_{1y}(\mathbb{a}) \\ g_{2x}(\mathbb{a}) & g_{2y}(\mathbb{a}) \end{pmatrix}$$

$$= (f_x(a)\ \ f_y(a))$$

$$- (f_x(a)\ f_y(a))\ \overbrace{D(G)^{-1}(a)\ D(G)(a)}^{I_2}$$

$$= (f_x(a)\ \ f_y(a)) - (f_x(a)\ \ f_y(a))$$

$$= \mathbf{0}$$

より以下の定理を得る.

<u>定理</u>　aが極大点ならば

$$\begin{cases} L_x(a) = L_y(a) = 0 \\ \lambda_1, \lambda_2 \geqq 0 \end{cases}$$

を満たす λ_1, λ_2 が存在する.