$$A = \begin{pmatrix} 6 & -3 & -7 \\ -1 & 2 & 1 \\ 5 & -3 & -6 \end{pmatrix}$$

1r + = 3r × (−1)

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda-6 & 3 & 7 \\ 1 & \lambda-2 & -1 \\ -5 & 3 & \lambda+6 \end{vmatrix} = \begin{vmatrix} \lambda-1 & 0 & 1-\lambda \\ 1 & \lambda-2 & -1 \\ -5 & 3 & \lambda+6 \end{vmatrix} = (\lambda-1)\begin{vmatrix} 1 & 0 & -1 \\ 1 & \lambda-2 & -1 \\ -5 & 3 & \lambda+6 \end{vmatrix}$$

$$= (\lambda-1)\begin{vmatrix} 1 & 0 & -1 \\ 0 & \lambda-2 & 0 \\ 0 & 3 & \lambda+1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-2 & 0 \\ 3 & \lambda+1 \end{vmatrix}$$

$= (\lambda-1)(\lambda-2)(\lambda+1)$ ⟶ 対角化できると分かる.

$\mathbb{R}^3$のAに対するスペクトル分解.

$$\mathbb{R}^3 = V(-1) \oplus V(1) \oplus V(2)$$

$\forall \vec{x} \in \mathbb{R}^3$

$\vec{x} = \vec{x}_1 + \vec{x}_2 + \vec{x}_3$ と書ける.

$\vec{x}_1 \in V(-1) \ni \vec{y}_1$

$\vec{x}_2 \in V(1) \ni \vec{y}_2$

$\vec{x}_3 \in V(2) \ni \vec{y}_3$

しかも一意的に書ける $\vec{x}_1 + \vec{x}_2 + \vec{x}_3 = \vec{y}_1 + \vec{y}_2 + \vec{y}_3 \Longrightarrow \vec{x}_i = \vec{y}_i \ (i=1,2,3)$

$A \in M_n(\mathbb{R})$

$\Phi_A(\lambda) = (\lambda-\alpha_1)\cdots(\lambda-\alpha_n)$

$\alpha_i \neq \alpha_j \ (i \neq j)$, $\alpha_i \in \mathbb{R}$

$\Longrightarrow$ Aは対角化可能

A : n次正方行列.　　$A\vec{x} = \alpha \vec{x}$

$$f(\lambda) = a_\ell \lambda^\ell + \cdots + a_1 \lambda + a_0$$

$$f(A) := a_\ell A^\ell + \cdots + a_1 A + a_0 I_n$$

$$\longrightarrow \quad f(A)\vec{x} = f(\alpha)\vec{x}.$$

$f_1(\lambda)$　　$f_1(-1) = 1,\ f_1(1) = 0,\ f_2(2) = 0$

$$f_1(\lambda) = \frac{(\lambda-1)(\lambda-2)}{(-1-1)(-1-2)} = \frac{1}{6}(\lambda-1)(\lambda-2)$$

Lagrange の補間多項式.

$$\vec{x} = \vec{x_1} + \vec{x_2} + \vec{x_3}$$

$$\begin{aligned}
f_1(A)\vec{x} &= f_1(A)\vec{x_1} + f_1(A)\vec{x_2} + f_1(A)\vec{x_3} \\
&= \underbrace{\boxed{f_1(-1)}}_{=1}\vec{x_1} + \underbrace{\boxed{f_1(1)}}_{=0}\vec{x_2} + \underbrace{\boxed{f_1(2)}}_{=0}\vec{x_3} \\
&= \vec{x_1}
\end{aligned}$$

$$P_1 = f_1(A) = \frac{1}{6}(A-I)(A-2I)$$

P

$f_2(\lambda)$ $\quad f_2(-1)=0,\ f_2(1)=1,\ f_2(2)=0$

$$\longrightarrow \quad f_2(\lambda) = \frac{(\lambda+1)(\lambda-2)}{(1+1)(1-2)} = -\frac{1}{2}(\lambda+1)(\lambda-2)$$

$$P_2 = f_2(A) = -\frac{1}{2}(A+I)(A-2I)$$

とおくと

$$f_2(A)\vec{x} = \vec{x_2}$$

$$f_3(\lambda) = \frac{(\lambda+1)(\lambda-1)}{(2+1)(2-1)} = \frac{1}{3}(\lambda+1)(\lambda-1)$$

$f_3(A) = \frac{1}{3}(A+I)(A-I)$ とおくと.

$$P_3 = f_3(A)\vec{x} = \vec{x_3}$$

$$P_1+P_2+P_3 = I \longleftarrow \vec{x} = P_1\vec{x} + P_2\vec{x} + P_3\vec{x}$$

$$= \vec{x_1} + \vec{x_2} + \vec{x_3}$$

$$g(\lambda) = f_1(\lambda) + f_2(\lambda) + f_3(\lambda) \equiv 1.$$

$$g(-1) = g(1) = g(2) = 1$$

$\alpha_i \neq \alpha_j \quad (i \neq j)$

$g(\alpha_1) = \cdots = g(\alpha_n) = g(\alpha_{n+1}) = 0$

g は n 次式以下

$\Longrightarrow \quad g \equiv 0$

$$A\vec{x} = A\vec{x_1} + A\vec{x_2} + A\vec{x_3}$$

$$= -\vec{x_1} + \vec{x_2} + 2\vec{x_3}$$

$$= -P_1\vec{x} + P_2\vec{x} + 2P_3\vec{x} = (-P_1 + 1.\,P_2 + 2P_3)\vec{x}$$

$$A = (-1)P_1 + 1\cdot P_2 + 2\cdot P_3$$

$$A^n = (-1)^n P_1 + 1^n P_2 + 2^n P_3.$$

$$P^{-1}AP = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 2 \end{pmatrix} \quad P^{-1}A^nP = \begin{pmatrix} (-1)^n & & \\ & 1^n & \\ & & 2^n \end{pmatrix} \quad A^n = P\begin{pmatrix} (-1)^n & & \\ & 1^n & \\ & & 2^n \end{pmatrix}P^{-1}$$

12/01 の前の週

$$A = \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$\Phi_A(\lambda) = \lambda^2(\lambda - 9)$$

1 次独立

$$\underset{\substack{\shortparallel \\ \ker(A)}}{V(0)} = \mathbb{R}\overbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}^{\vec{q}_1} + \mathbb{R}\overbrace{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}^{\vec{q}_2}$$

$$V(9) = \mathbb{R}\underbrace{\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}}_{\vec{q}_3}$$

$$\mathbb{R}^3 = V(0) \oplus V(9)$$

$$\boxed{\begin{array}{l} \mathbb{R}^n \supset V \supset W \quad \text{部分空間.} \\ \dim V = \dim W \Longrightarrow V = W \end{array}}$$

$$\boxed{\begin{array}{l} \alpha \neq \beta \\ A\vec{x}_1 = \alpha \vec{x}_1 \\ A\vec{x}_2 = \beta \vec{x}_2 \\ \vec{x}_1 + \vec{x}_2 = \vec{0} \\ \Longrightarrow \vec{x}_1 = \vec{x}_2 = 0 \end{array}}$$

$$\mathbb{R}^3 \supset V(0) + V(9)$$

3 次元 　　3 次元

$$\longrightarrow \mathbb{R}^3 = V(0) + V(9)$$

$V(0)+V(9) \ni \vec{x}$

$\vec{x} = \vec{x}_1 + \vec{x}_2 \quad \vec{x}_1 \in V(0),\ \vec{x}_2 \in V(9)$

$= c_1\vec{q}_1 + c_2\vec{q}_2 + c_3\vec{q}_3$

1) $V(0)+V(9)$ は $\vec{q}_1, \vec{q}_2, \vec{q}_3$ で生成される。

2) $c_1\vec{q}_1 + c_2\vec{q}_2 + c_3\vec{q}_3 = \vec{0}$ $\longrightarrow$ $\begin{cases} c_1\vec{q}_1 + c_2\vec{q}_2 = \vec{0} \\ c_3\vec{q}_3 = \vec{0} \end{cases}$

$\underbrace{c_1\vec{q}_1 + c_2\vec{q}_2}_{\in V(0)} + \underbrace{c_3\vec{q}_3}_{\in V(9)}$

$\vec{q}_1 \nparallel \vec{q}_2$

$\rightarrow c_1 = c_2 = 0$

$\downarrow \vec{q}_3 \neq \vec{0}$

$c_3 = 0$

$\longrightarrow$ $\vec{q}_1, \vec{q}_2, \vec{q}_3$ は1次独立.

1), 2)
$\rightsquigarrow$ $V(0)+V(1)$ の基底

$\vec{q}_1, \vec{q}_2, \vec{q}_3$ は

$\dim\left(V(0)+V(9)\right) = 3$

$\mathbb{R}^3 = V(0) \oplus V(9)$ 　$\mathbb{R}^3$ の A によるスペクトル分解

$\vec{x} = \vec{x_1} + \vec{x_2}$

$f_1(\lambda)$ 　$f_1(0) = 1,\ f_1(9) = 0$

$$= \frac{(\lambda - 9)}{0 - 9} = -\frac{1}{9}(\lambda - 9)$$

$f_2(\lambda)$ 　$f_2(0) = 0,\ f_2(9) = 1$

$$= \frac{\lambda}{9}$$

$\vec{x_1} = -\frac{1}{9}(A - 9I)\vec{x}$

$\vec{x_2} = \frac{1}{9} A \vec{x}$

まとめ

$A \in M_n(\mathbb{R})$ 　$\Phi_A(\lambda) = \prod_{j=1}^{\ell} (\lambda - \alpha_j)^{m_j}$

$m_j \geq 1$ 　$\alpha_i \neq \alpha_j \quad (i \neq j)$

$\alpha_j \in \mathbb{R}$

(1) A が対角化可能

$\Longleftrightarrow$ (2) $\mathbb{R}^n = V(\alpha_1) \oplus \cdots \oplus V(\alpha_\ell)$

$\Longleftrightarrow$ (3) $\varphi(\lambda) = (\lambda - \alpha_1) \cdots (\lambda - \alpha_\ell)$ とすると

$$\varphi(A) = O_n$$

$P_1 = f_1(A)$

$$f_1(\lambda) = \frac{(\lambda - \alpha_2) \cdots (\lambda - \alpha_\ell)}{(\alpha_1 - \alpha_2) \cdots (\alpha_1 - \alpha_\ell)}$$

A : 3次実対称行列.
$= (a_{ij})$

$$\Phi_A(\lambda) = \lambda^3 - (a_{11}+a_{22}+a_{33})\lambda^2 + *\lambda - |A|.$$

対称でなくとも OK.

A : 実対称行列.

定理 $\Phi_A(\alpha) = 0 \Longrightarrow \alpha \in \mathbb{R}$.

代数学の基本定理 $f(\lambda) = \lambda^n + a_{n-1}\lambda + \cdots + a_1\lambda + a_0$ $\quad a_j \in \mathbb{C}$

$\Longrightarrow f(\lambda) = (\lambda-\alpha_1)\cdots(\lambda-\alpha_n)$ と因数分解できる.

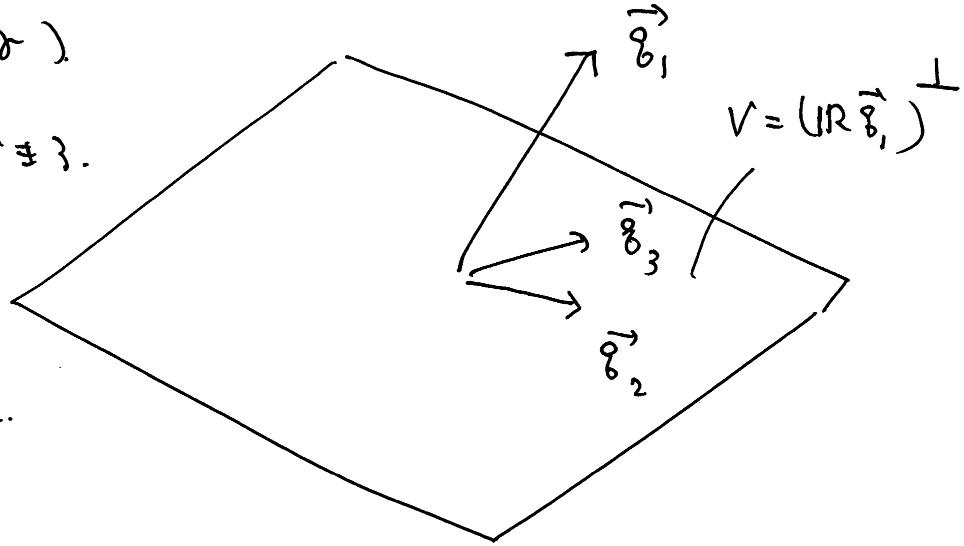
$\alpha_j \in \mathbb{C}$

$\Phi_A(\lambda) = (\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)$.

$A\vec{q}_1 = \alpha\vec{q}_1$, $\|\vec{q}_1\| = 1$ とできる.

Vの正規直交基底.

$\vec{q}_1, \vec{q}_2, \vec{q}_3$ は $\mathbb{R}^3$ の正規直交基底.



$Q = (\vec{q}_1\ \vec{q}_2\ \vec{q}_3)$ は直交.

$\mathbb{R}^3$

$AQ = (A\vec{q}_1\ \ A\vec{q}_2\ \ A\vec{q}_3)$ ⟨ $\vec{q}_1, \vec{q}_2, \vec{q}_3$ は基底.

$= (\alpha\vec{q}_1\ \ \underset{\substack{\shortparallel \\ 0}}{*}\vec{q}_1 + *\vec{q}_2 + *\vec{q}_3\ \ \text{———})$

$= (\vec{q}_1\ \vec{q}_2\ \vec{q}_3)\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{B}} \\ 0 & & \end{pmatrix}$

$(A\vec{q}_2, \vec{q}_1) = (\vec{q}_2, \underbrace{{}^tA}_{=A}\vec{q}_1)$

$= (\vec{q}_2, A\vec{q}_1) = (\vec{q}_2, \alpha\vec{q}_1)$

$= \alpha(\vec{q}_2, \vec{q}_1) = 0$

$= (*_1\vec{q}_1 + *_2\vec{q}_2 + *_3\vec{q}_3, \vec{q}_1)$

$= *_1$

$\underbrace{{}^tQ}_{Q^{-1}}\ AQ = \overbrace{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & B & \\ 0 & & \end{pmatrix}}^{対称.}$

$\shortparallel$

${}^t({}^tQAQ) = \begin{pmatrix} \alpha & \\ & {}^tB \end{pmatrix}$

$\shortparallel$

${}^tQ\,{}^tA\,{}^t({}^tQ)$

$= {}^tQAQ$

A対称.

Bは対称.

$B = {}^tB$

${}^t(ABC)$
$= {}^tC\,{}^tB\,{}^tA$

Bは回転行列で対角化できる.

$${}^tR_1 B R_1 = \begin{pmatrix} \beta' & 0 \\ 0 & \gamma' \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & R_1 & \end{pmatrix}$$ ← 直交になる.

$${}^tQAQ = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & & \\ 0 & B & \end{pmatrix}$$

$${}^tR\,{}^tQAQ\,R = \begin{pmatrix} 1 & \\ & {}^tR_1 \end{pmatrix}\begin{pmatrix} \alpha & \\ & B \end{pmatrix}\begin{pmatrix} 1 & \\ & R_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & \\ & {}^tR_1BR_1 \end{pmatrix} = \begin{pmatrix} \alpha & & \\ & \beta' & \\ & & \gamma' \end{pmatrix}$$

$$= {}^t(QR)A(QR)$$

QRは直交.

A_j: 正方.

$$\begin{pmatrix} \alpha & \\ & A_1 \end{pmatrix}\begin{pmatrix} \beta & \\ & A_2 \end{pmatrix} = \begin{pmatrix} \alpha\beta & \\ & A_1A_2 \end{pmatrix}$$

Q_1, Q_2 直交

$\Rightarrow Q_1Q_2$ 直交

I　Q_1, Q_2 直交 $\Longrightarrow$ $Q_1 Q_2$ 直交を示せ.

$${}^tQ_j Q_j = Q_j\, {}^tQ_j = I_n \qquad j = 1, 2.$$

II　A : 正方行列 実対称, 正則. ${}^tA = A$

A^{-1} は対称を示せ

ただし　$A \cdot A^{-1} = A^{-1} \cdot A = I_n$

転置をとる

${}^tI = I.$