ポートフォリオ理論

資産	A_1	A_2	$\cdots$	A_n
収益率	R_1	R_2	$\cdots$	R_n
1期	$R_{1,1}$	$R_{2,1}$	$\cdots$	$R_{n,1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
N期	$R_{1,N}$	$R_{2,N}$	$\cdots$	$R_{n,N}$
投資比率	x_1	x_2	$\cdots$	x_n

1単位の資金を投入.

$\longrightarrow x_1 + \cdots + x_n = 1$

ポートフォリオの収益率.

$$R = x_1 R_1 + \cdots + x_n R_n \longrightarrow \mu := \overline{R} = x_1 \overline{R}_1 + \cdots + x_n \overline{R}_n$$

$$\mu_j = \overline{R}_j \qquad = x_1 \mu_1 + \cdots + x_n \mu_n.$$

$$\left.\begin{aligned} g_1 &= 1 - x_1 - \cdots - x_n = 0 \\ g_2 &= \mu - x_1\mu_1 - \cdots - x_n\mu_n = 0 \end{aligned}\right\} \text{の下で} \quad f(\vec{x}) = \frac{1}{2} V(R) = \frac{1}{2}(V\vec{x}, \vec{x})$$

$V = (\sigma_{ij})$ $\quad \sigma_{ij} = \begin{cases} V(R_j) & i = j \\ \mathrm{cov}(R_i, R_j) & i \neq j. \end{cases}$

$= {}^tDD$ 対称.

$D = (\vec{R}_1 \cdots \vec{R}_n)$

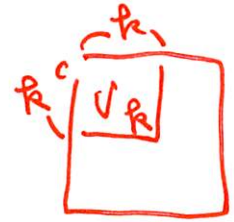
$$\vec{R}_j = \frac{1}{\sqrt{N}} \begin{pmatrix} R_{j,1} - \bar{R}_j \\ \vdots \\ R_{j,N} - \bar{R}_j \end{pmatrix}$$

仮定

① $\vec{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \nparallel \vec{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

② $(V\vec{x}, \vec{x}) > 0 \quad (\vec{x} \neq \vec{0})$ $\iff |V| > 0,\ |V_{n-1}| > 0, \cdots$

$\cdots, |V_1| > 0$



k次
主座小行列.

$\iff$ V の固有値

$\alpha_1, \cdots, \alpha_n > 0$.

$\iff$ $\vec{R}_1, \cdots, \vec{R}_n$ は1次独立.

証明.

$\underbrace{c_1\vec{R}_1 + \cdots + c_n\vec{R}_n}_{= D\vec{c}} = \vec{0} \implies c_1 = \cdots = c_n = 0$

Lagrange の未定乗数法で解く.

$$\nabla(f) + \lambda_1 \nabla(g_1) + \lambda_2 \nabla(g_2) = \vec{0}$$

$$\rightarrow \quad V\vec{x} + \lambda_1 \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -\mu_1 \\ \vdots \\ -\mu_n \end{pmatrix} = \vec{0}$$

A 対称.

$$f(\vec{x}) = \frac{1}{2}(A\vec{x}, \vec{x})$$

$$\rightarrow \nabla(f) = A\vec{x}$$

$$V\vec{x} = \lambda_1 \vec{1} + \lambda_2 \vec{\mu}$$

$\rightarrow$

② から V 正則より

$$\vec{x} = \lambda_1 V^{-1}\vec{1} + \lambda_2 V^{-1}\vec{\mu}$$

$$= V^{-1}(\vec{1}\ \vec{\mu})\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

<u>λ_1, λ_2 を求める.</u>

制約条件より

$$1 = (\vec{1}, \vec{x})$$

$$\mu = (\vec{\mu}, \vec{x})$$

$$\begin{pmatrix} 1 \\ \mu \end{pmatrix} = \underbrace{\begin{pmatrix} {}^t\vec{1} \\ {}^t\vec{\mu} \end{pmatrix}\vec{x}}_{\begin{pmatrix} {}^t\vec{1}\vec{x} \\ {}^t\vec{\mu}\vec{x} \end{pmatrix}} = {}^t(\vec{1}\ \vec{\mu})\, V^{-1}(\vec{1}\ \vec{\mu})\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} {}^t\vec{1} \\ {}^t\vec{\mu} \end{pmatrix}\left(V^{-1}\vec{1}\ \ V^{-1}\vec{\mu}\right)\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{\begin{matrix} {}^t\vec{1}V^{-1}\vec{1} \\ {}^t\vec{\mu}V^{-1}\vec{1} \end{matrix}} & \begin{matrix} \boxed{{}^t\vec{1}V^{-1}\vec{\mu}} \\ \boxed{{}^t\vec{\mu}V^{-1}\vec{\mu}} \end{matrix} \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$\left({}^tF\, V^{-1} F \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right) = \left(V^{-1} F \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, F \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right)$$

$$= \left(V^{-1}(c_1 \vec{1} + c_2 \vec{\mu}), (c_1 \vec{1} + c_2 \vec{\mu})\right)$$

$F = (\vec{1}\ \vec{\mu})$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq \vec{0} \xrightarrow[①]{} c_1 \vec{1} + c_2 \vec{\mu} \neq \vec{0}$$

> 0

V^{-1}は正定値.

$$\begin{pmatrix} 1 \\ \mu \end{pmatrix} = \begin{pmatrix} C & A \\ A & B \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$ において $$\begin{vmatrix} C & A \\ A & B \end{vmatrix} = BC - A^2 > 0$$

$= D$.

参2 215p.

$$\lambda_1 = \frac{\begin{vmatrix} 1 & A \\ \mu & B \end{vmatrix}}{D} = \frac{B - A\mu}{D}$$

$$\lambda_2 = \frac{\begin{vmatrix} C & 1 \\ A & \mu \end{vmatrix}}{D} = \frac{C\mu - A}{D}.$$

λ_1, λ_2 は参2計算と逆

$F = (\vec{1}\ \vec{\mu})$

まとめ

$$\begin{pmatrix}1\\ \mu\end{pmatrix} = \boxed{\begin{pmatrix}{}^t\vec{1}V^{-1}\vec{1} & {}^t\vec{1}V^{-1}\vec{\mu}\\ {}^t\vec{\mu}V^{-1}\vec{1} & {}^t\vec{\mu}V^{-1}\vec{\mu}\end{pmatrix}}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}$$

$$= {}^tFV^{-1}F\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \qquad = \begin{pmatrix}C & A\\ A & B\end{pmatrix}$$

対称.

$$\boxed{\begin{aligned}{}^t({}^tFV^{-1}F) &= {}^tF\,{}^t(V^{-1})\,{}^t({}^tF)\\ &= {}^tFV^{-1}F\end{aligned}}$$

V^{-1}: 対称 $\leftarrow$ V 対称.

${}^t(V^{-1}) = V^{-1}$

key point ${}^tFV^{-1}F$ 2×2 対称.

正定値.

$\longrightarrow |{}^tFV^{-1}F| > 0$

V: 正定値. $\exists P$: 直交

$${}^tPVP = \begin{pmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_n\end{pmatrix}$$

$\alpha_j > 0$

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$({}^tPVP)^{-1} = (P^{-1}VP)^{-1} = P^{-1}V^{-1}(P^{-1})^{-1}$

$= P^{-1}V^{-1}P = {}^tPV^{-1}P$

$${}^tPV^{-1}P = \begin{pmatrix}\frac{1}{\alpha_1} & & \\ & \ddots & \\ & & \frac{1}{\alpha_n}\end{pmatrix}$$

V^{-1}が定める2次形式は

正定値.

$$\vec{x} = \lambda_1 V^{-1} \vec{1} + \lambda_2 V^{-1} \vec{\mu}$$

$$= V^{-1} (\vec{1} \ \vec{\mu}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$= \frac{B - A\mu}{D} V^{-1} \vec{1} + \frac{C\mu - A}{D} V^{-1} \vec{\mu}$$ の場合

$$\therefore \ V(R) = (V\vec{x}, \vec{x})$$

$$= \left((\vec{1} \ \vec{\mu}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \ V^{-1} (\vec{1} \ \vec{\mu}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \ {}^t(\vec{1} \ \vec{\mu}) V^{-1} (\vec{1} \ \vec{\mu}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} C & A \\ A & B \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} 1 \\ \mu \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \right) = \lambda_1 + \mu \lambda_2 .$$

$$= \frac{B - A\mu}{D} + \mu \frac{C\mu - A}{D} = \frac{C\mu^2 - 2A\mu + B}{D}$$

$$V(R) = \sigma^2 = \frac{C\mu^2 - 2A\mu + B}{D}$$

グラフを描く.

$$= \frac{C\left(\mu - \frac{A}{C}\right)^2 + B - \frac{A^2}{C}}{D}$$

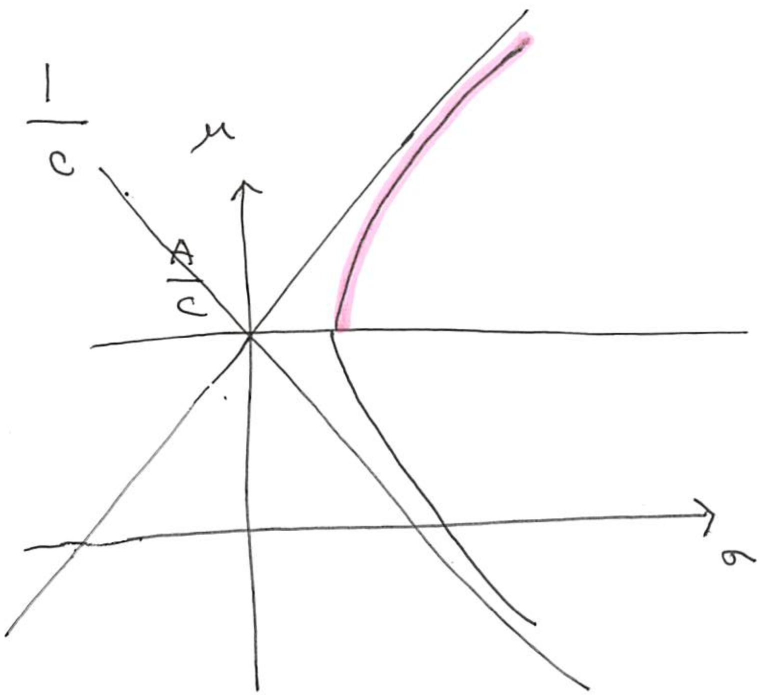
$$= \frac{C^2\left(\mu - \frac{A}{C}\right)^2 + \boxed{\underbrace{BC - A^2}_{=D}}}{CD}$$

$$= \frac{C\left(\mu - \frac{A}{C}\right)^2}{D} + \frac{1}{C}$$

$$\frac{\sigma^2}{\frac{1}{C}} - \frac{C^2\left(\mu - \frac{A}{C}\right)^2}{D} = 1$$

$$\boxed{\frac{\sigma^2}{\frac{1}{C}} - \frac{\left(\mu - \frac{A}{C}\right)^2}{\frac{D}{C^2}} = 1}$$

文の部分が
フロンティア.



スペクトル分解.

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{pmatrix}$$

$$1r \mathrel{+}= 3r$$

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda-1 & -2 & 2 \\ -1 & \lambda-1 & -1 \\ -1 & 2 & \lambda-4 \end{vmatrix} = \begin{vmatrix} \lambda-2 & 0 & \lambda-2 \\ -1 & \lambda-1 & -1 \\ -1 & 2 & \lambda-4 \end{vmatrix}$$

$$= (\lambda-2)\begin{vmatrix} 1 & 0 & 1 \\ -1 & \lambda-1 & -1 \\ -1 & 2 & \lambda-4 \end{vmatrix} = (\lambda-2)\begin{vmatrix} 1 & 0 & 1 \\ 0 & \lambda-1 & 0 \\ 0 & 2 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-2)\begin{vmatrix} \lambda-1 & 0 \\ 2 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda-3)$$

$A : 3 \times 3$ $\quad \Phi_A(\lambda) = (\lambda-\alpha_1)(\lambda-\alpha_2)(\lambda-\alpha_3)$

$\alpha_i \neq \alpha_j \ (i \neq j) \Longrightarrow$ Aは対角化できる.

※証明は来週.

定理 (2) $A\vec{v}_1 = \alpha\vec{v}_1$, $\boxed{A\vec{v}_2 = \beta\vec{v}_2}$, $\alpha \neq \beta$

$(A-\beta I)\vec{v}_2 = \vec{0}$

A: $n \times n$.

$\vec{v}_1 + \vec{v}_2 = \vec{0} \Longrightarrow \vec{v}_1 = \vec{v}_2 = \vec{0}$

$(A-\gamma I)\vec{v}_3 = \vec{0}$

(3) $A\vec{v}_1 = \alpha\vec{v}_1$, $A\vec{v}_2 = \beta\vec{v}_2$, $\boxed{A\vec{v}_3 = \gamma\vec{v}_3}$ $\alpha \neq \beta$, $\beta \neq \gamma$, $\gamma \neq \alpha$

$\Longrightarrow \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0} \Longrightarrow \vec{v}_1 = \vec{v}_2 = \vec{v}_3 = \vec{0}$

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(2) $\vec{v}_1 + \vec{v}_2 = \vec{0} \xrightarrow{(A-\beta I)\cdot} (\alpha-\beta)\vec{v}_1 = \vec{0}$

$\xrightarrow[\alpha\neq\beta]{} \vec{v}_1 = \vec{0} \xrightarrow[\vec{v}_1+\vec{v}_2=\vec{0}]{} \vec{v}_2 = \vec{0}$

(3) $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0} \xrightarrow[(A-\gamma I)\cdot]{} \underbrace{\boxed{(\alpha-\gamma)\vec{v}_1}}_{=\vec{w}_1} + \underbrace{\boxed{(\beta-\gamma)\vec{v}_2}}_{=\vec{w}_2} = \vec{0}$

$A\vec{w}_1 = \alpha\vec{w}_1 \xrightarrow[(2)]{} \vec{w}_1 = \vec{w}_2 = \vec{0} \xrightarrow[\vec{w}_1+\vec{w}_2+\vec{w}_3=\vec{0}]{} \vec{w}_3 = \vec{0}$.

$A\vec{w}_1 = A(\alpha-\gamma)\vec{v}_1 = (\alpha-\gamma)A\vec{v}_1 = (\alpha-\gamma)\alpha\vec{v}_1 = \alpha\vec{w}_1$

$A\vec{w}_2 = \beta\vec{w}_2$

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{pmatrix}$$ の固有ベクトルを求める.