$$A = \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{rank}(A) = 1$

$\ker(A) = \{ \vec{v} \in \mathbb{R}^3 ; A\vec{v} = \vec{0} \}$

$\dim \ker(A) = 3 - 1 = 2.$

3: 変数の個数　　1: $\mathrm{rank}(A)$

[illegible]

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda-4 & -4 & 2 \\ -4 & \lambda-4 & 2 \\ 2 & 2 & \lambda-1 \end{vmatrix}$$

$$\| \quad |\lambda I_3 - A|$$

$$= \begin{vmatrix} \lambda & -\lambda & 0 \\ -4 & \lambda-4 & 2 \\ 2 & 2 & \lambda-1 \end{vmatrix}$$

$$= \lambda \begin{vmatrix} 1 & -1 & 0 \\ -4 & \lambda-4 & 2 \\ 2 & 2 & \lambda-1 \end{vmatrix}$$

$$= \lambda \begin{vmatrix} 1 & -1 & 0 \\ 0 & \lambda-8 & 2 \\ 0 & 4 & \lambda-1 \end{vmatrix}$$

$$= \lambda \begin{vmatrix} \lambda-8 & 2 \\ 4 & \lambda-1 \end{vmatrix}$$

$-$ $+$

$$= \lambda \{ (\lambda-8)(\lambda-1) - 8 \} = \lambda(\lambda^2 - 9\lambda) = \lambda^2(\lambda-9)$$

$\leadsto \Phi_A(0) = 0$

$$\| \quad |0 I_3 - A| = |-\vec{a}_1 \ -\vec{a}_2 \ -\vec{a}_3|$$

$$= (-1)^3 |\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3|$$

$$= -|A|$$

A：正方行列

$|A| = 0 \iff \exists \vec{v} \neq \vec{0}, \ A\vec{v} = \vec{0}$

固有ベクトルを求める.

$\underline{\lambda = 0}$

$(0 I_3 - A)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0} \iff A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0} \iff x + y - \frac{1}{2}z = 0$

$\Updownarrow$

$\left(\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = 0$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y + \frac{1}{2}z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \frac{z}{2}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\vec{q}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \vec{q}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \longrightarrow$ 1次独立 ∵

$(c_1\vec{q}_1 + c_2\vec{q}_2 = \vec{0} \Longrightarrow c_1 = c_2 = 0)$

$\parallel$

$\begin{pmatrix} * \\ c_1 \\ 2c_2 \end{pmatrix} = \vec{0} \rightsquigarrow c_1 = c_2 = 0.$

$\vec{q}_1, \vec{q}_2$ は $V(0)$ の基底.

① $V(0)$ を生成

② 1次独立

$\forall \vec{v} \in V(0) \quad \vec{v} = c_1\vec{q}_1 + c_2\vec{q}_2$

に表せる

$\underline{\lambda = 9}$

$(9 I_3 - A)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0}$

$\iff \boxed{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 4 & 8 \end{pmatrix}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0} \iff \begin{cases} x \quad\; + 2z = 0 \\ \quad y + 2z = 0 \end{cases}$

$\rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

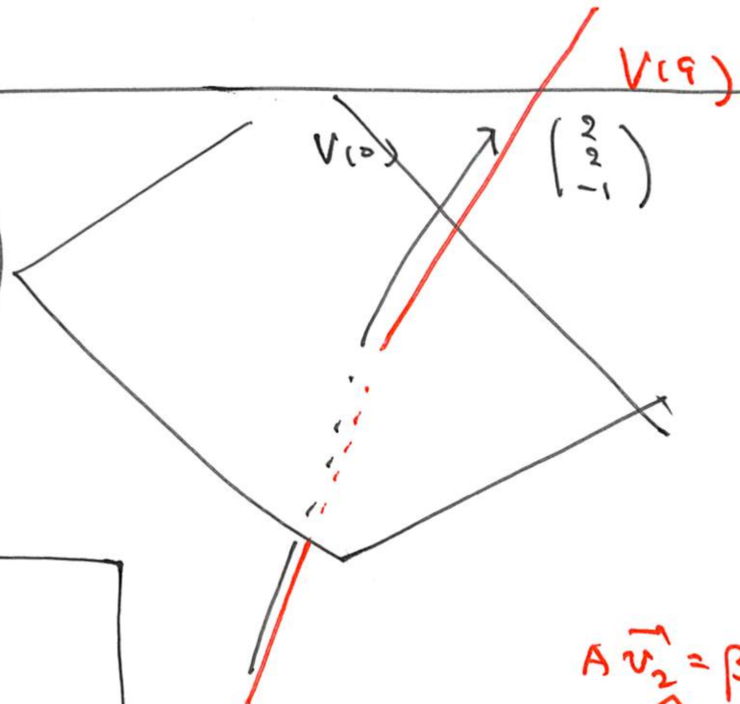
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2z \\ -2z \\ z \end{pmatrix} = z\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \quad (z \neq 0)$$ が固有ベクトル

A：対称行列　$V(\alpha) \perp V(\beta)$

$\alpha \neq \beta$

$$\vec{q}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{q}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \vec{q}_3 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

は1次独立.



補題　$\alpha \neq \beta$.　A：n次正方行列

$\vec{v}_1 \in V(\alpha)$, $\boxed{\vec{v}_2 \in V(\beta)}$

$\vec{v}_1 + \vec{v}_2 = \vec{0} \implies \vec{v}_1 = \vec{v}_2 = 0$

$$A\vec{v}_2 = \beta\vec{v}_2$$
$$\Updownarrow$$
$$\boxed{(A - \beta I_n)\vec{v}_2 = \vec{0}}$$

$$\vec{0} = (A - \beta I_n)(\vec{v}_1 + \vec{v}_2) = (A - \beta I_n)\vec{v}_1 = A\vec{v}_1 - \beta\vec{v}_1$$

$$= \alpha\vec{v}_1 - \beta\vec{v}_1 = \underbrace{(\alpha - \beta)}_{\neq 0}\vec{v}_1 \rightsquigarrow \vec{v}_1 = \vec{0} \rightsquigarrow \vec{v}_2 = \vec{0}$$

$(A - \beta I_n)\vec{v}_2 = \vec{0}$

$$\underbrace{c_1\vec{q}_1 + c_2\vec{q}_2}_{\in V(0)} + \underbrace{c_3\vec{q}_3}_{\in V(9)} = \vec{0} \quad \leadsto \quad c_1 = c_2 = c_3 = 0$$

To be shown

À montrer

$$\xrightarrow[\text{補足}]{} \quad c_1\vec{q}_1 + c_2\vec{q}_2 = \vec{0}, \quad c_3\vec{q}_3 = \vec{0} \quad \longrightarrow \quad c_1 = c_2 = 0, \quad c_3 = 0$$

$\vec{q}_1, \vec{q}_2$ は1次独立

$\vec{q}_3 \neq 0$

$Q = (\vec{q}_1\ \vec{q}_2\ \vec{q}_3)$ 3×3.

Q 正則 $\iff |Q| \neq 0 \iff \vec{q}_1, \vec{q}_2, \vec{q}_3$ は1次独立

$\left(c_1\vec{q}_1 + c_2\vec{q}_2 + c_3\vec{q}_3 = \vec{0} \implies c_1 = c_2 = c_3 = 0 \right)$

$\to Q$: 正則

$$AQ = (A\vec{q}_1\ A\vec{q}_2\ A\vec{q}_3) = (0\vec{q}_1\ 0\vec{q}_2\ 9\vec{q}_3)$$

$$= (\vec{q}_1\ \vec{q}_2\ \vec{q}_3)\begin{pmatrix} 0 & & \\ & 0 & \\ & & 9 \end{pmatrix} \longrightarrow Q^{-1}AQ = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 9 \end{pmatrix}$$

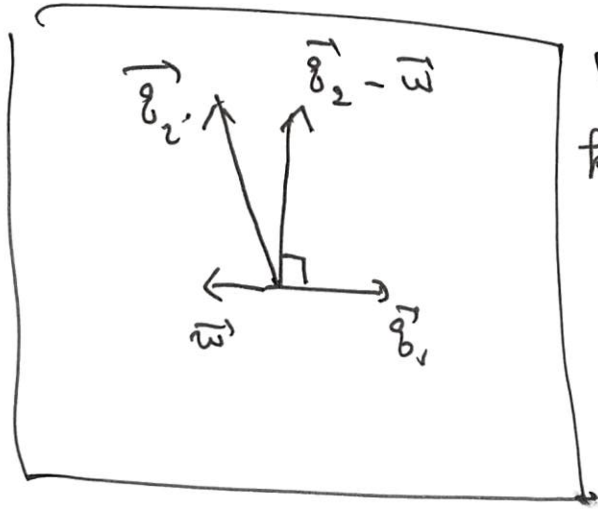
と対角化.

Aを直交行列で対角化する。

$V(0)$　$\vec{q}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{q}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ が基底。

$\vec{w}$　$\vec{q}_2$ の $\vec{q}_1$ 方向への直交射影

$$= \frac{(\vec{q}_1, \vec{q}_2)}{\|\vec{q}_1\|^2}\vec{q}_1 = \frac{-1}{2}\vec{q}_1$$



$V(0)$
$\shortparallel$
$\ker(A)$

$$\vec{p}_1 = \frac{1}{\|\vec{q}_1\|}\vec{q}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{p}_2 = \frac{1}{\|\vec{q}_2 - \vec{w}\|}(\vec{q}_2 - \vec{w}) = \frac{1}{3\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$\vec{q}_3 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

$$\vec{q}_2 - \vec{w} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$\vec{p}_3 = \frac{1}{3}\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

$\|\vec{p}_1\| = \|\vec{p}_2\| = \|\vec{p}_3\| = 1$, $(\vec{p}_1, \vec{p}_2) = 0$

A: 対称, aとて $\alpha \neq \beta$

$V(\alpha) \perp V(\beta)$

$\longrightarrow$

$(\vec{p}_1, \vec{p}_3) = (\vec{p}_2, \vec{p}_3) = 0$

$P = (\vec{p}_1\ \vec{p}_2\ \vec{p}_3)$ は直交行列。

$$(P\vec{v}, P\vec{w}) = (\vec{v}, \vec{w}) \quad (\forall \vec{v}, \vec{w} \in \mathbb{R}^3)$$

$$\Longleftrightarrow \quad {}^tPP = I_3$$

$$\Longleftrightarrow \quad {}^tPP = P\,{}^tP = I_3$$

$\Longleftrightarrow$ $P = (\vec{p_1}\ \vec{p_2}\ \vec{p_3})$ とすると $\vec{p_1}, \vec{p_2}, \vec{p_3}$ は正規直交系.

$$\left(A\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \qquad A = \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$\parallel$$

$$4x^2 + 4y^2 + z^2 + 8xy - 4xz - 4yz.$$

$$= \left({}^tPA\begin{pmatrix} x \\ y \\ z \end{pmatrix}, {}^tP\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \left({}^tPAP\cdot{}^tP\begin{pmatrix} x \\ y \\ z \end{pmatrix}, {}^tP\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)$$

^tP は直交

$$= \left(\begin{pmatrix} 0 & & \\ & 0 & \\ & & 9 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}\right) = 9\zeta^2.$$

$$AP = P\begin{pmatrix} 0 & & \\ & 0 & \\ & & 9 \end{pmatrix} \longrightarrow {}^tPAP = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 9 \end{pmatrix} \qquad \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = {}^tP\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$^tP = P^{-1}$

P は直交だから

以下は 11/03 のノート.

ポートフォリオ.

資産 $A_1, A_2, \cdots, A_n$.

収益率 $R_1, R_2, \cdots, R_n$

1日目 $R_{1,1}, R_{2,1}, \cdots, R_{n,1}$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

N日目 $R_{1,N}, R_{2,N}, \cdots, R_{n,N}$

投資 $x_1 \quad x_2 \quad \cdots \quad x_n$

ポートフォリオ

$$x_1 + x_2 + \cdots + x_n - 1 = 0$$

$$R = x_1 R_1 + \cdots + x_n R_n.$$

$\mu_j = \overline{R_j}$ とすると

$$\mu = \overline{R} = x_1 \overline{R_1} + \cdots + x_n \overline{R_n}$$

$$= x_1 \mu_1 + \cdots + x_n \mu_n$$

$$g_1 = 1 - x_1 - \cdots - x_n = 0$$

$$g_2 = \mu - x_1\mu_1 - \cdots - x_n\mu_n = 0$$

の下で

$$(\vec{1}, \vec{x}) = 1$$

$$(\vec{\mu}, \vec{x}) = \mu.$$

$$f = \frac{1}{2} \cdot V(R) = \frac{1}{2}(V\vec{x}, \vec{x})$$

$$V = (\sigma_{ij}) \qquad \sigma_{ij} = \begin{cases} V(R_j) & i = j \\ \mathrm{cov}(R_i, R_j) & i \neq j. \end{cases}$$

$$= {}^t D D \qquad D = (\vec{R}_1\ \vec{R}_2 \cdots \vec{R}_n)$$

$$\vec{R}_j = \frac{1}{\sqrt{N}} \begin{pmatrix} R_{j,1} - \overline{R}_j \\ \vdots \\ R_{j,N} - \overline{R}_j \end{pmatrix}$$

仮定 ① $\vec{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \nparallel \vec{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

② $(V\vec{x}, \vec{x}) > 0 \quad (\vec{x} \neq \vec{0})$

$\Leftrightarrow$ $|V| > 0,\ |V_{n-1}| > 0, \cdots$

$\cdots |V_2| > 0,\ |V_1| > 0.$

$$\nabla(f) + \lambda_1 \nabla(g_1) + \lambda_2 \nabla(g_2) = \vec{0}$$

$$V\vec{x} + \lambda_1 \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -\mu_1 \\ \vdots \\ -\mu_n \end{pmatrix} = \vec{0}$$

$\longrightarrow$ $V\vec{x} = \lambda_1 \vec{1} + \lambda_2 \vec{\mu}$ を満たす λ_1, λ_2 が存在.

$$\boxed{\begin{array}{l} V\text{は正則.} \\ ②\text{の仮定より } |V| > 0 \end{array}}$$

$$\begin{aligned} \vec{x} &= V^{-1}(\lambda_1 \vec{1} + \lambda_2 \vec{\mu}) \\ &= \lambda_1 V^{-1}\vec{1} + \lambda_2 V^{-1}\vec{\mu} \\ &= \boxed{(V^{-1}\vec{1} \;\; V^{-1}\vec{\mu})} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \\ &= V^{-1}(\vec{1}\;\vec{\mu}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 1 \\ \mu \end{pmatrix} = \begin{pmatrix} (\vec{1}, \vec{x}) \\ (\vec{\mu}, \vec{x}) \end{pmatrix} = \begin{pmatrix} {}^t\vec{1} \\ {}^t\vec{\mu} \end{pmatrix} \vec{x} = \underbrace{\boxed{\begin{pmatrix} {}^t\vec{1} \\ {}^t\vec{\mu} \end{pmatrix} V^{-1} (\vec{1}\;\vec{\mu})}}_{= {}^tF V^{-1} F} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} {}^t\vec{1} V^{-1} \vec{1} & {}^t\vec{1} V^{-1}\vec{\mu} \\ {}^t\vec{\mu} V^{-1} \vec{1} & {}^t\vec{\mu} V^{-1}\vec{\mu} \end{pmatrix}$$

とする.

$${}^t\vec{\mu}\, V^{-1} \vec{1} = (\vec{\mu}, V^{-1}\vec{1}) = ({}^t(V^{-1})\vec{\mu}, \vec{1}) = (V^{-1}\vec{\mu}, \vec{1})$$

$$= (\vec{1}, V^{-1}\vec{\mu}) = {}^t\vec{1}\, V^{-1}\vec{\mu}$$

2行より

(V：対称, 正則 → V^{-1} 対称.)

= 1行2より

$$F = (\vec{1}\ \vec{\mu}) \qquad {}^tF\, V^{-1} F = \begin{pmatrix} {}^t\vec{1} \\ {}^t\vec{\mu} \end{pmatrix} V^{-1} (\vec{1}\ \vec{\mu})$$

$${}^t({}^tF\, V^{-1} F) = {}^tF\, {}^t(V^{-1})\, {}^t({}^tF)$$

$$= {}^tF\, V^{-1} F$$

V^{-1}：対称.

${}^tF\, V^{-1} F$ は対称.

$$\boxed{\begin{aligned} &{}^t(AB) \\ &= {}^tB\, {}^tA \end{aligned}}$$

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$$\begin{pmatrix} C & A \\ A & B \end{pmatrix} \text{〃}\ {}^tF\, V^{-1} F$$

$\begin{vmatrix} C & A \\ A & B \end{vmatrix} > 0$ がわかる.

$$\left({}^tFV^{-1}F\begin{pmatrix}c_1\\c_2\end{pmatrix}, \begin{pmatrix}c_1\\c_2\end{pmatrix}\right)$$

$\begin{pmatrix}c_1\\c_2\end{pmatrix} \neq \vec{0}$ ならば >0.

$$= \left(V^{-1}F\begin{pmatrix}c_1\\c_2\end{pmatrix}, F\begin{pmatrix}c_1\\c_2\end{pmatrix}\right)$$

$F = (\vec{\imath}\ \vec{\mu})$

$$= \left(V^{-1}(c_1\vec{\imath}+c_2\vec{\mu}), (c_1\vec{\imath}+c_2\vec{\mu})\right) > 0$$

$\begin{pmatrix}c_1\\c_2\end{pmatrix} \neq \vec{0}$

① $\vec{\imath} \nparallel \vec{\mu}$ より $\begin{pmatrix}c_1\\c_2\end{pmatrix} \neq \vec{0} \longrightarrow c_1\vec{\imath}+c_2\vec{\mu} \neq \vec{0}$.

② V^{-1} は正値 $\longrightarrow$ 2次形式は正定値. $(V^{-1}\vec{x}, \vec{x}) > 0$ $(\vec{x} \neq \vec{0})$

$|{}^tFV^{-1}F| > 0$, $c > 0$

$$\| \quad \begin{vmatrix} C & A \\ A & B \end{vmatrix}$$

$$\left(\begin{pmatrix}a & c\\ c & e\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) > 0$$

$\begin{pmatrix}x\\y\end{pmatrix} \neq \vec{0}$

$$\Longleftrightarrow \begin{vmatrix}a & c\\ c & e\end{vmatrix} > 0,\ a > 0$$

②について　V: 正定値　$\exists P$: 直交

$$ {}^tPVP = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \quad \alpha_1, \cdots, \alpha_n > 0 $$

$$ P^{-1}VP \quad " $$

$$ \rightarrow \quad P^{-1}V^{-1}P = \begin{pmatrix} \frac{1}{\alpha_1} & & \\ & \ddots & \\ & & \frac{1}{\alpha_n} \end{pmatrix} $$

$$ \boxed{\begin{aligned} &(ABC)^{-1} \\ &= C^{-1}B^{-1}A^{-1} \\ &(A^{-1})^{-1} = A \end{aligned}} $$

V^{-1}の

$\rightarrow$ 固有値は $\frac{1}{\alpha_1}, \cdots, \frac{1}{\alpha_n} > 0$.

$\rightarrow \quad (V^{-1}\vec{x}, \vec{x}) > 0 \quad (\vec{x} \neq \vec{0})$

$$ \begin{pmatrix} 1 \\ \mu \end{pmatrix} = \begin{pmatrix} C & A \\ A & B \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} $$ が解ける.

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{pmatrix}$$ に対して $\Phi_A(x)$ を計算.