2015/11/10.

$$A = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

$$f(x, y, z) = \frac{1}{2}\left(A\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)$$

$$= ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz.$$

$$\nabla(f) = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2ax + 2dy + 2ez \\ 2dx + 2by + 2fz \\ 2ex + 2fy + 2cz \end{pmatrix}$$

$$= \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

---

一般に　$A : n \times n$ 対称行列.

$$f(x_1, \cdots, x_n) = \frac{1}{2}\left(A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right)$$

$$\nabla(f) = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

II $A$ : $n \times n$ 対称. ${}^tA = A$.

$B$ : $n \times 2$ ⇝ ${}^tB$ $2 \times n$

---

$C = {}^tB A B$ は対称.　　　${}^t(AB) = {}^tB\,{}^tA$.

${}^tC = {}^tB\,{}^tA\,{}^t({}^tA) = {}^tB A A = C$

より $C$ は対称.

# 対称行列の対角化.

**n=2**　　A: 2×2 対称.

Ⅰ $\Phi_A(\lambda) = |\lambda I_2 - A| = (\lambda-\alpha)(\lambda-\beta)$

$\alpha, \beta \in \mathbb{R}$

Ⅱ ∃ R: 回転行列. $R^{-1}AR = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ とできる.

Ⅲ $\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \alpha\xi^2 + \beta\eta^2$

但し $\begin{pmatrix} x \\ y \end{pmatrix} = R\begin{pmatrix} \xi \\ \eta \end{pmatrix}$

$= \left(R^{-1}AR \cdot R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}, R^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right)$

$R^{-1}$: 回転

$= \alpha\xi^2 + \beta\eta^2$

R 回転

$\Longrightarrow (R\vec{v}, R\vec{w}) = (\vec{v}, \vec{w})$

$(\vec{v}, \vec{w} \in \mathbb{R})$

注 逆は成立しない.

Ⅳ $\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) > 0 \quad \left(\begin{pmatrix}x\\y\end{pmatrix} \neq \vec{0}\right)$

$<$

$\Longleftrightarrow \alpha, \beta > 0 \Longleftrightarrow$ $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ とすると

$<$

$a > 0, \quad |A| > 0.$

$<$

---

$A$: $n \times n$ 実対称行列.

Ⅰ (証明はしない) $\Phi_A(\lambda) = |\lambda I_n - A|$

$= (\lambda - \alpha_1) \cdots (\lambda - \alpha_n)$

$= \lambda^n - (a_{11} + \cdots + a_{nn})\lambda^{n-1} + \cdots + (-1)^n |A|$

$\Longrightarrow \lambda_j \in \mathbb{R} \quad (j = 1, \ldots, n)$

Ⅱ $\exists P$: 直交行列. $\Longleftrightarrow (P\vec{v}, P\vec{w}) = (\vec{v}, \vec{w})$ $\rightsquigarrow$ 更に深めていく

$(\vec{v}, \vec{w} \in \mathbb{R}^n)$

${}^tP =$ $P^{-1}AP = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$

III $\left(A\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix}, \begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix}\right) = \alpha_1\xi_1^2 + \cdots + \alpha_n\xi_n^2$

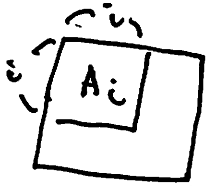
$$\vec{x} = P\vec{\xi} \longleftrightarrow \vec{\xi} = P^{-1}\vec{x}$$

$$\underset{\substack{\uparrow \\ P^{-1}\text{直交}}}{=} \left(P^{-1}A\vec{x}, P^{-1}\vec{x}\right) = \left(P^{-1}AP\cdot P^{-1}\vec{x}, P^{-1}\vec{x}\right)$$

$$= \left(\begin{pmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_n\end{pmatrix}\vec{\xi}, \vec{\xi}\right)$$

$$= \alpha_1\xi_1^2 + \cdots + \alpha_n\xi_n^2$$

IV $(A\vec{x}, \vec{x}) > 0 \quad (\vec{x} \neq \vec{0}) \overset{(a)}{\Longleftrightarrow} \alpha_1, \cdots, \alpha_n > 0$




i次主座小行列

$$\overset{(c)}{\Longleftrightarrow} |A_i| > 0 \quad (i = 1, \cdots, n)$$

## 直交行列

$P : n\times n$. 実

$P$ が 直交 $\iff (P\vec{v}, P\vec{w}) = (\vec{v}, \vec{w})$ ＃

$(\vec{v}, \vec{w} \in \mathbb{R}^n)$

＃ $\iff (({}^tPP - I_n)\vec{v}, \vec{w}) = \vec{0} \quad (\vec{v}, \vec{w} \in \mathbb{R}^n)$

$\iff ({}^tPP - I_n)\vec{v} = \vec{0} \quad (\vec{v} \in \mathbb{R}^n)$

$\iff {}^tPP - I_n = O_n$

$\iff {}^tPP = I_n$

$\vec{a} \in \mathbb{R}^n$

$(\vec{a}, \vec{v}) = 0 \quad (\vec{v} \in \mathbb{R}^n)$

$\Updownarrow$

$\vec{a} = \vec{0}$

$\Downarrow \quad \vec{v} = \vec{a}$

$(\vec{a}, \vec{a}) = 0$

$\|\vec{a}\|^2$

$\iff \vec{a} = \vec{0}$

$A : m \times n$

$A\vec{v} = \vec{0} \quad (\vec{v} \in \mathbb{R}^n)$

$\iff A = O_{m,n}$

$\Leftarrow$

$\vec{0} = (\vec{a}_1 \cdots \vec{a}_n)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \vec{e}_j = \vec{a}_j$

$A\vec{e}_j = \vec{a}_j$

$P$が直交 $\iff (P\vec{v}, P\vec{w}) = (\vec{v}, \vec{w}) \quad (\vec{v}, \vec{w} \in \mathbb{R}^n)$

$\iff {}^tPP = I_n$

$\iff {}^tPP = P{}^tP = I_n.$

$\iff \|\vec{p}_j\| = 1$

$(\vec{p}_i, \vec{p}_j) = 0 \quad (i \neq j)$

両辺 $|\ |$

$$\boxed{\begin{array}{l} |{}^tPP| = |I_n| = 1 \\ \quad \parallel \\ |{}^tP|\cdot|P| = |P|^2 \qquad |{}^tP| = |P| \end{array}}$$

$\rightsquigarrow |P| = \pm 1 \neq 0.$

$P$：直交 $\rightsquigarrow$ $P$：正則

${}^tPP = I_n$

$\underset{\cdot P^{-1}}{\rightsquigarrow} ({}^tP\cdot P)P^{-1} = I_nP^{-1} = P^{-1}$

$\parallel$

${}^tP.$

$\boxed{{}^tP = P^{-1}}$

$\rightsquigarrow P{}^tP = P\cdot P^{-1} = I_n$

${}^tPP = I_n$ $\qquad$ $P = (\vec{p}_1\ \vec{p}_2\ \vec{p}_3)$

<u>$n=3.$</u>

$$
{}^tPP = \begin{pmatrix} {}^t\vec{p}_1 \\ {}^t\vec{p}_2 \\ {}^t\vec{p}_3 \end{pmatrix} (\vec{p}_1\ \vec{p}_2\ \vec{p}_3)
$$

$$
= \begin{pmatrix} {}^t\vec{p}_1\vec{p}_1 & {}^t\vec{p}_1\vec{p}_2 & {}^t\vec{p}_1\vec{p}_3 \\ {}^t\vec{p}_2\vec{p}_1 & {}^t\vec{p}_2\vec{p}_2 & {}^t\vec{p}_2\vec{p}_3 \\ {}^t\vec{p}_3\vec{p}_1 & {}^t\vec{p}_3\vec{p}_2 & {}^t\vec{p}_3\vec{p}_3 \end{pmatrix} = \begin{pmatrix} \|\vec{p}_1\|^2 & & \\ (\vec{p}_2, \vec{p}_1) & \|\vec{p}_2\|^2 & \\ (\vec{p}_3, \vec{p}_1) & & \|\vec{p}_3\|^2 \end{pmatrix} = I_3.
$$

$(\vec{p}_3, \vec{p}_2)$

$$
\boxed{{}^t\vec{a}\,\vec{b} = (\vec{a}, \vec{b})}
$$

$$
{}^tPP = I_n \rightsquigarrow \|\vec{p}_j\| = 1,\quad (\vec{p}_i, \vec{p}_j) = 0 \ (i \neq j)
$$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ 回転 直交 である.

$$(R\vec{v}, R\vec{w}) = (\vec{v}, \vec{w})$$

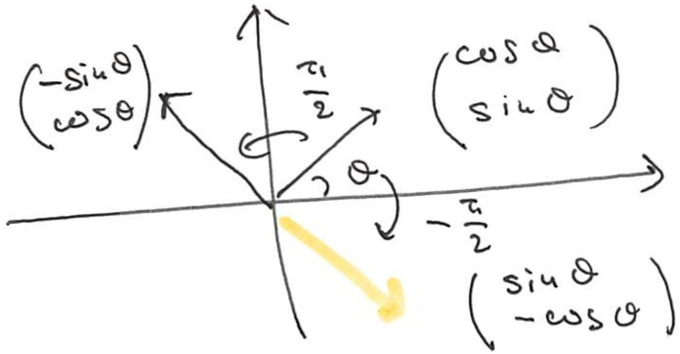
$$R^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = {}^tR$$

$$R^{-1} = {}^tR \rightsquigarrow {}^tRR = I_2 \quad \text{ok.}$$

---

$$|R| = \cos^2\theta + \sin^2\theta = 1$$

$$Q = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \rightsquigarrow$$ 直交 $$|Q| = -\cos^2\theta - \sin^2\theta = -1.$$



$P$: 直交 $\Longrightarrow$ ${}^tP = P^{-1}$ は直交

$\Updownarrow$

${}^tPP = I_n$

$\Updownarrow$

${}^tPP = P\,{}^tP = I_n$

${}^t({}^tP)\cdot{}^tP = P\cdot{}^tP = I_n$

より ${}^tP$ は直交.

---

IV $(a) \Longrightarrow (b)$

$P^{-1}AP = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$

$AP = P\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \longleftrightarrow (A\vec{p}_1 \cdots A\vec{p}_j \cdots A\vec{p}_n)$

$= (\alpha_1\vec{p}_1 \cdots \alpha_j\vec{p}_j \cdots \alpha_n\vec{p}_n)$

$A\vec{p}_j = \alpha\vec{p}_j \quad (j = 1, \cdots, n)$ $\quad \|\vec{p}_j\| = 1$ より

$\vec{p}_j \neq \vec{0}$

$(A\vec{p}_j, \vec{p}_j) > 0$ $\nwarrow (a)$

$\|\vec{p}\| = 1$

$\downarrow$

$= (\alpha_j\vec{p}_j, \vec{p}_j) = \alpha_j\|\vec{p}_j\|^2 = \alpha_j$

$\alpha_j > 0.$

(e) $\Rightarrow$ (a)

$$(A\vec{x}, \vec{x}) = \overset{\overset{0}{\wedge\!|}}{\alpha_1 \xi_1^2} + \cdots + \overset{\overset{0}{\wedge\!|}}{\alpha_n \xi_n^2} \geq 0$$

$\Updownarrow$

$\alpha_1, \cdots, \alpha_n > 0$

$$\boxed{\begin{array}{l} c_1, \cdots, c_n \geq 0 \\ c_1 + \cdots + c_n = 0 \\ \Rightarrow c_1 = \cdots = c_n = 0 \end{array}}$$

$= \;\Leftrightarrow \alpha_1 \xi_1^2 = \cdots = \alpha_n \xi_n^2 = 0$

$\Leftrightarrow \xi_1 = \cdots = \xi_n = 0$

$\Leftrightarrow \vec{x} = \vec{0}$

$\quad \shortparallel \quad P\vec{\xi} = P\vec{0} = \vec{0}$

$\rightarrow \left( \vec{x} \neq \vec{0} \Rightarrow (A\vec{x}, \vec{x}) > 0 \right)$ (a)

(a) ($\Leftrightarrow$ (e)) を仮定

$\alpha_1, \cdots, \alpha_n > 0$

$$P^{-1}AP = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$$

$$\left.\begin{array}{l} |P^{-1}AP| = \alpha_1 \cdots \alpha_n > 0 \\ \quad \shortparallel \\ |P^{-1}| \cdot |A| \cdot |P| = |A| \end{array}\right) \rightarrow |A| > 0$$

$= |P^{-1}P| = |I_n| = 1$

$$A = \begin{pmatrix} 3 & -1 & -2 \\ -1 & 3 & 2 \\ -2 & 2 & 6 \end{pmatrix}$$ に対し

$\Phi_A(\lambda)$ を計算.

時間があるときは各固有値の固有ベクトルを求めよ.