emath 2015
10/27

$g(x, y, z) = x + y + z - 1 = 0$ の下で

$$w = f(x, y, z) = x^2 + 4y^2 + 9z^2$$

(x, y, z) で極値をとるとする

$$\begin{cases} x + y + z - 1 = 0 & \cdots ① \quad \leftarrow \text{制約条件.} \\ \nabla(f) + \lambda \nabla(g) = \vec{0} & \cdots ② \end{cases}$$

を満たす $\lambda \in \mathbb{R}$ が存在する.

$$\nabla(f) = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

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② $\Leftrightarrow$ $\begin{pmatrix} 2x \\ 8y \\ 18z \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \vec{0}$

$\Leftrightarrow$ $x = -\frac{1}{2}\lambda$, $y = -\frac{1}{8}\lambda$, $z = -\frac{1}{18}\lambda$.

を①に代入すると

$$-\frac{1}{2}\lambda - \frac{1}{8}\lambda - \frac{1}{18}\lambda - 1 = 0$$

$$= -\lambda\left(\frac{1}{2} + \frac{1}{8} + \frac{1}{18}\right)$$

$$= -\lambda \frac{36 + 9 + 4}{72}$$

$$= -\lambda \frac{49}{72}$$

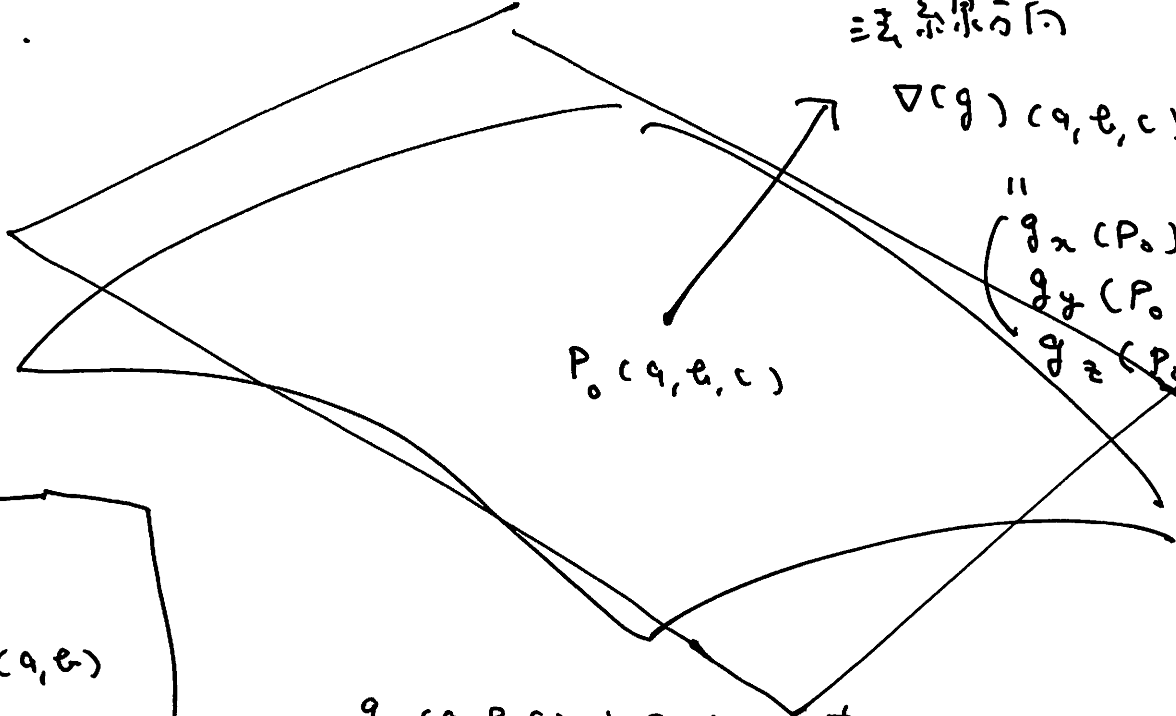
より　$\lambda = -\frac{72}{49}$ となるので

$$x = \frac{36}{49}, \quad y = \frac{9}{49}, \quad z = \frac{4}{49}$$

3変数と制約条件。

$g(x, y, z) = 0$

曲面。



法線方向

$$\nabla(g)(a, b, c) = \begin{pmatrix} g_x(P_0) \\ g_y(P_0) \\ g_z(P_0) \end{pmatrix}$$

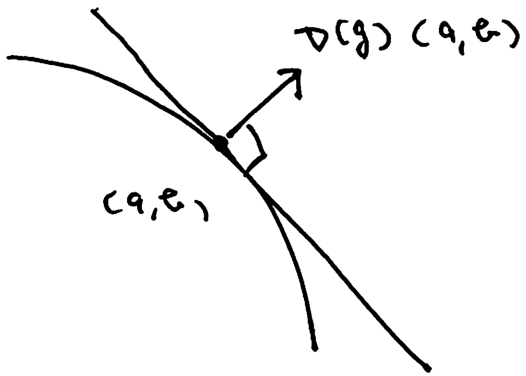
$g_z(a, b, c) \neq 0$ と仮定。

$\longrightarrow$ 陰関数の定理

(a, b, c) の近くで

$z = \varphi(x, y)$

$g(x, y) = 0$



接平面

$$z = \varphi_x(a,b)(x-a) + \varphi_y(a,b)(y-b) + \underset{\varphi(a,b)}{\underset{\parallel}{c}}$$

$$g(x, y, \varphi(x,y)) \equiv 0 \quad ((a,b)\text{の近くで})$$

$$g_x(x, y, \varphi(x,y)) \cdot 1 + g_y(x, y, \varphi(x,y)) \cdot 0 +$$
$$g_z(\quad) \cdot \frac{\partial \varphi}{\partial x}(x,y) = 0$$

(a,b) を代入

$$g_x(a,b,c) + g_z(a,b,c)\frac{\partial \varphi}{\partial x}(a,b) = 0$$

同様に

$$g_y(a,b,c) + g_z(a,b,c)\frac{\partial \varphi}{\partial y}(a,b) = 0$$

$$z = -\frac{g_x(P_0)}{g_z(P_0)}(x-a) - \frac{g_y(P_0)}{g_z(P_0)}(y-b) + c$$

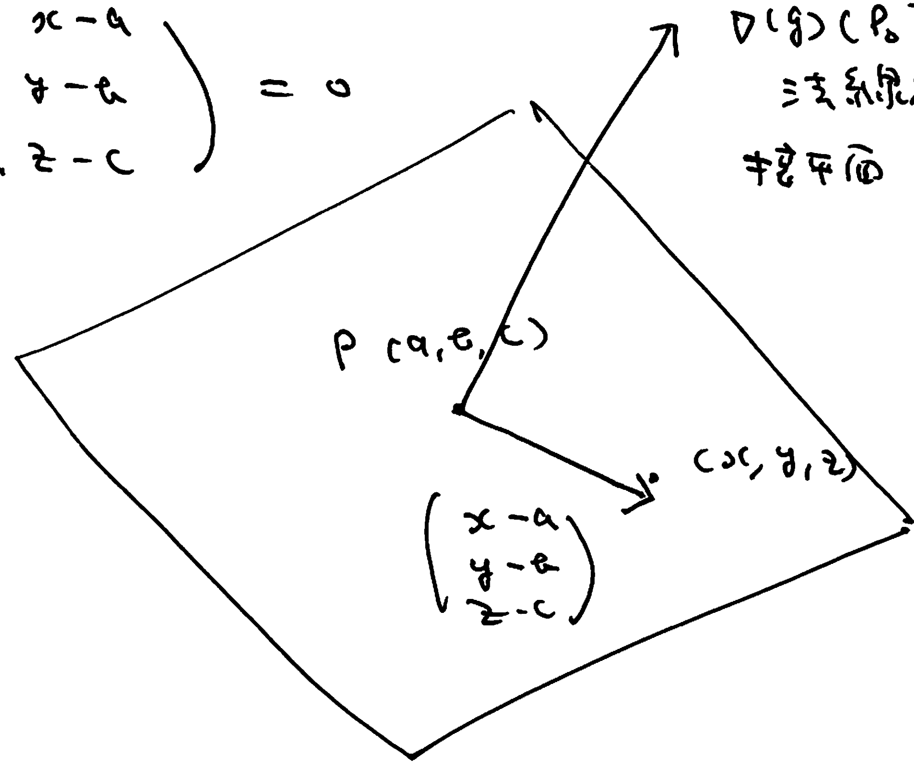
$$\longrightarrow g_x(P_0)(x-a) + g_y(P_0)(y-b) + g_z(P_0)(z-c) = 0$$

$$\begin{pmatrix} g_x(P_0) \\ g_y(P_0) \\ g_z(P_0) \end{pmatrix} \cdot \begin{pmatrix} x-a \\ y-b \\ z-c \end{pmatrix} = 0$$

$\nabla(g)(P_0)$

法線方向.

接平面



$$\begin{cases} g_1(x,y,z)=0 \\ g_2(x,y,z)=0 \end{cases}$$ の下で $w = f(x,y,z)$

(CTの334pあたりの方法と違う

$$g_1(a,b,c) = g_2(a,b,c) = 0.$$

$$\nabla g_1(P_0) \nparallel \nabla(g_2)(P_0)$$

とする

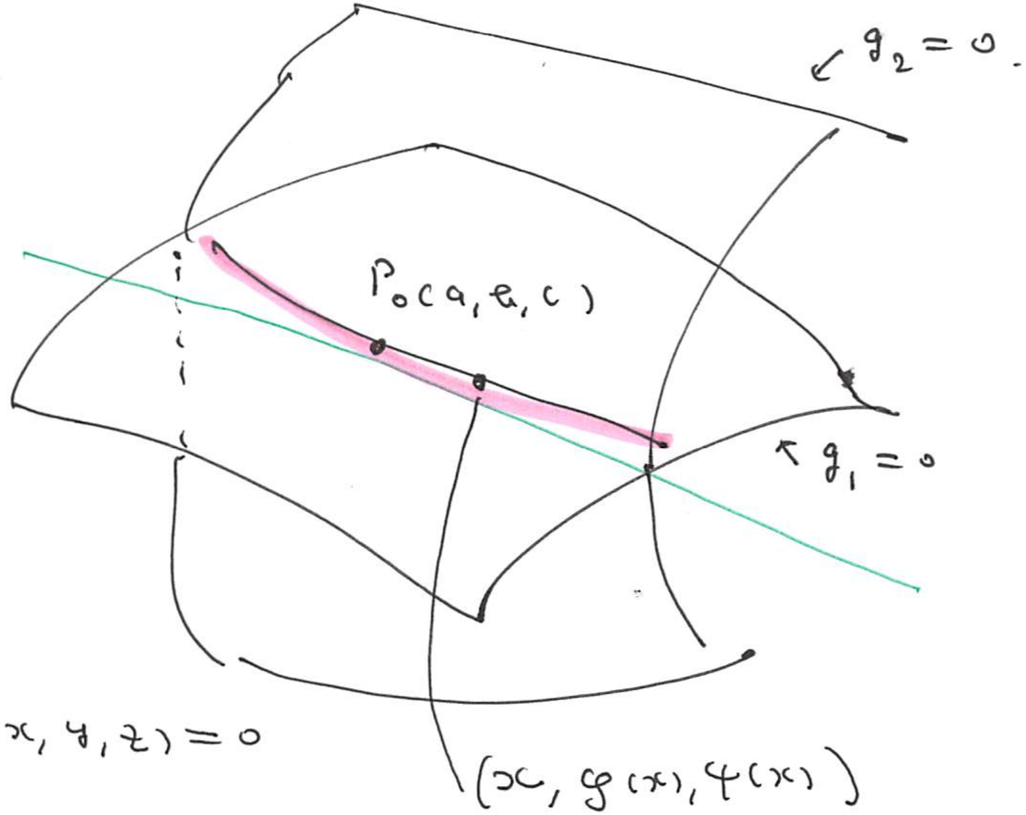
$$\begin{vmatrix} g_{1y}(P_0) & g_{2y}(P_0) \\ g_{1z}(P_0) & g_{2z}(P_0) \end{vmatrix} \neq 0$$

の場合を考える.

→ 陰関数定理 (a,b,c) の近くで $g_1(x,y,z) = g_2(x,y,z) = 0$

$$y = \varphi(x),\quad z = \psi(x)$$

と書ける



$$\vec{v_1} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \quad \vec{v_2} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\vec{v_1} \nparallel \vec{v_2} \iff \left(c_1\vec{v_1} + c_2\vec{v_2} = \vec{0} \implies c_1 = c_2 = 0 \right)$$

否定 $\vec{v_1} \parallel \vec{v_2} \iff c_1\vec{v_1} + c_2\vec{v_2} = \vec{0}$ で $c_1 \neq 0$ or $c_2 \neq 0$

$\rightsquigarrow$ $c_1 \neq 0$ のとき $\vec{v_2} = -\dfrac{c_1}{c_2}\vec{v_1}$

$c_2 \neq 0$ のとき $\vec{v_1} = -\dfrac{c_2}{c_1}\vec{v_2}$

$$\iff \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \neq 0 \text{ or } \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} \neq 0$$

$\uparrow$

省略.

$$\text{or } \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \neq 0$$

$F(t) = f(t, \varphi(t), \psi(t))$ を考える.

(a, b, c) で極値を持つ $\rightarrow F'(a) = 0$

Chain Rule

$$F(t) = f(x(t), y(t), z(t))$$

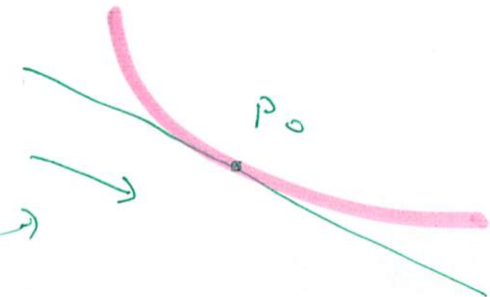
$$\boxed{\begin{aligned} F'(t) = &\, f_x(\quad)\cdot x'(t) \\ &+ f_y(\quad)\cdot y'(t) \\ &+ f_z(\quad)\cdot z'(t) \end{aligned}}$$

$$\begin{aligned} F'(t) &= f_x(\quad)\cdot 1 \\ &\quad + f_y(\quad)\cdot \varphi'(t) \\ &\quad + f_z(\quad)\cdot \psi'(t) \end{aligned}$$

$\longrightarrow$ $t = a$ を代入

$$f_x(P_0) + f_y(P_0)\varphi'(a) + f_y(P_0)\psi'(a) = 0$$

$$\Longleftrightarrow \quad \left(\nabla(f)(P_0), \begin{pmatrix} 1 \\ \varphi'(a) \\ \psi'(a) \end{pmatrix}\right) = 0$$



$$g_1(t, \varphi(t), \psi(t)) \equiv 0, \quad g_2(t, \varphi(t), \psi(t)) \equiv 0$$

$$g_{1x}(\quad)\cdot 1 + g_{1y}(\quad)\cdot \varphi'(t) + g_{1z}(\quad)\cdot \psi'(t) \equiv 0$$

2 2 2

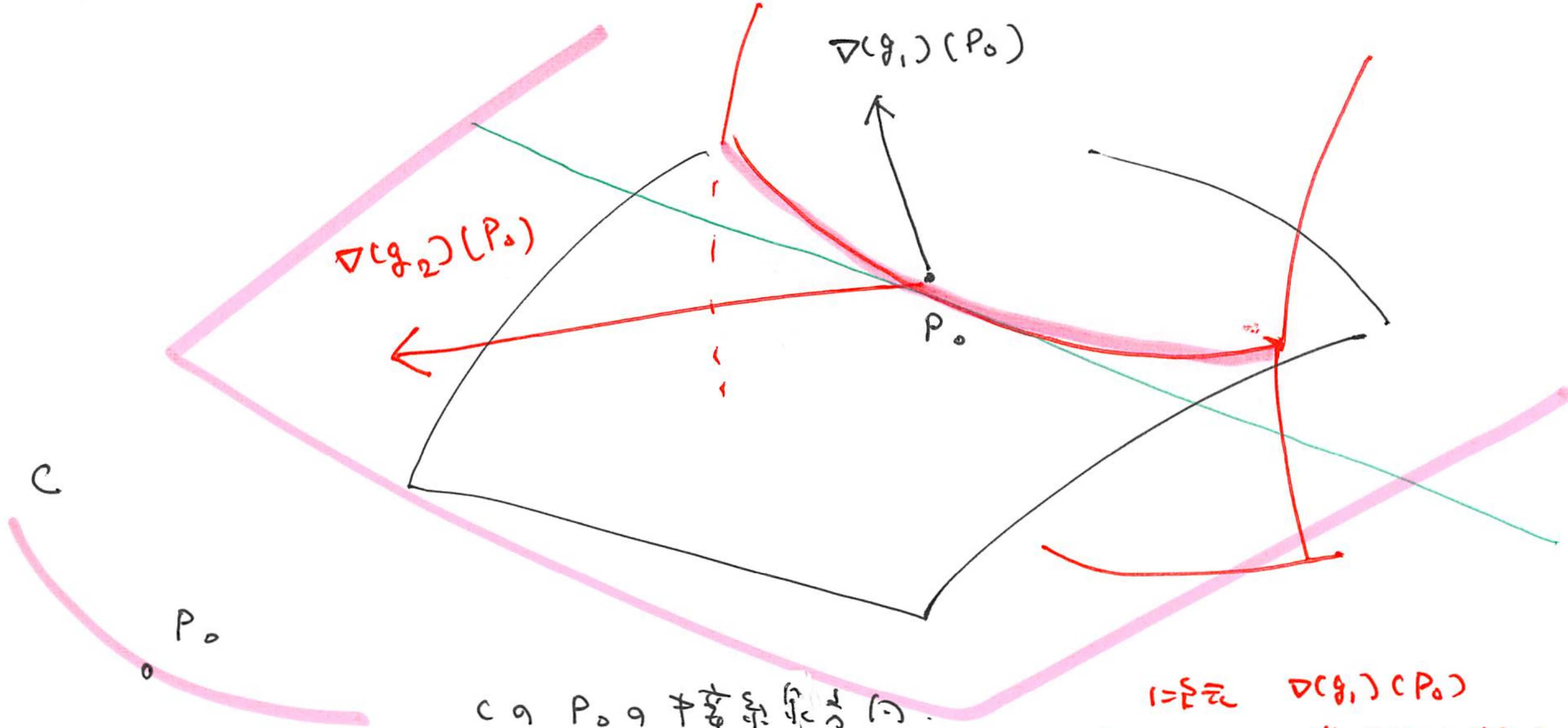
$t=a$ とすれば

$$g_{1x}(P_0) + g_{1y}(P_0)\varphi'(a) + g_{1z}(P_0)\psi'(a) = 0$$

2 2 2

$$\Longleftrightarrow \quad \left(\nabla(g_1)(P_0), \begin{pmatrix} 1 \\ \varphi'(a) \\ \psi'(a) \end{pmatrix} \right) = 0$$

$$\left(\nabla(g_2)(P_0), \begin{pmatrix} 1 \\ \varphi'(a) \\ \psi'(a) \end{pmatrix} \right) = 0$$

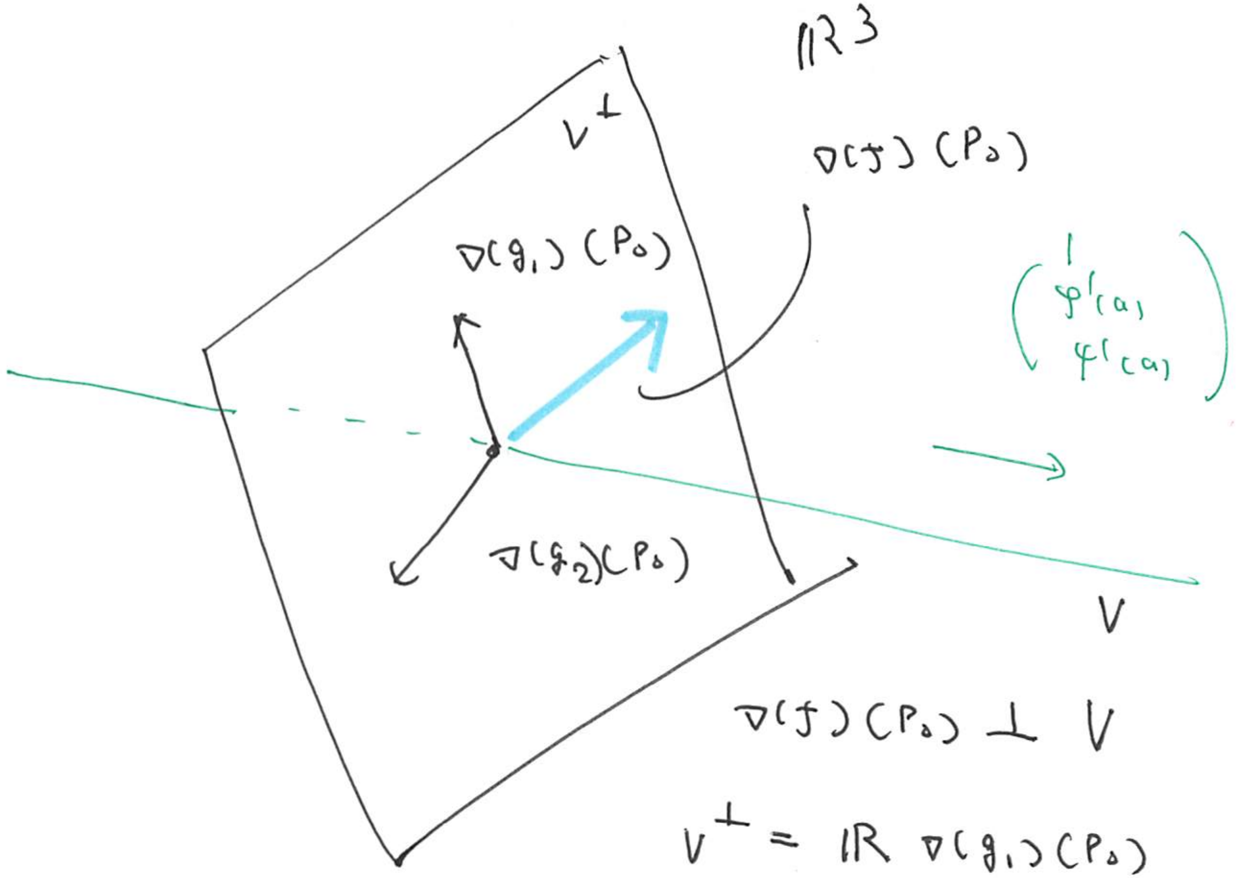


C の P_0 の接線方向

仮定 $\nabla(g_1)(P_0) \nparallel \nabla(g_2)(P_0)$

$$\begin{pmatrix} 1 \\ \varphi'(a) \\ \psi'(a) \end{pmatrix} \in V = \left\{ \vec{v} \in \mathbb{R}^3 ;\ \begin{array}{l} (\vec{v}, \nabla(g_1)(P_0)) = 0 \\ (\vec{v}, \nabla(g_2)(P_0)) = 0 \end{array} \right\} = \mathbb{R} \begin{pmatrix} 1 \\ \varphi'(a) \\ \psi'(a) \end{pmatrix}$$

$$\left(\nabla(f)(P_0), \begin{pmatrix} 1 \\ \varphi'(a) \\ \psi'(a) \end{pmatrix} \right) = 0$$



$\nabla(f)(P_0) \perp V$

$$V^\perp = \mathbb{R}\,\nabla(g_1)(P_0) + \mathbb{R}\,\nabla(g_2)(P_0)$$

$$\longrightarrow \quad \nabla(f)(P_0) + \lambda_1 \nabla(g_1)(P_0) + \lambda_2 \nabla(g_2)(P_0) = \vec{0}$$

を満たす $\lambda_1, \lambda_2 \in \mathbb{R}$ が存在.

問題　$g_1(x, y, z) = g_2(x, y, z) = 0$ の下で

$$w = f(x, y, z)$$

を考える

定理　$P_0(a, b, c)$ で極値をとる

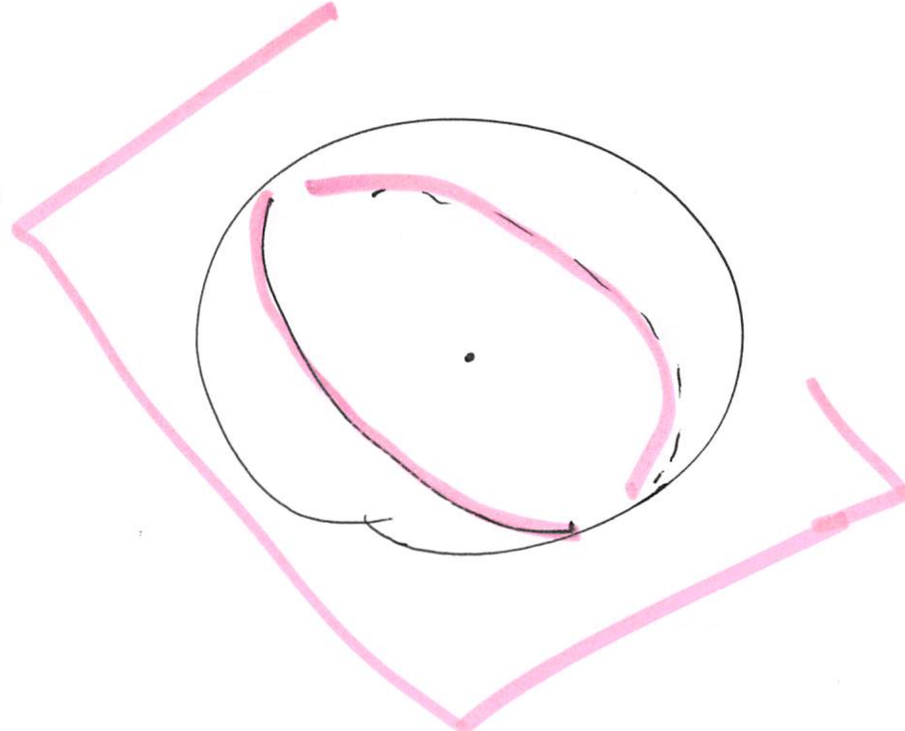
$$\Rightarrow \begin{cases} g_1(P_0) = g_2(P_0) = 0 \\ \nabla(f)(P_0) + \lambda_1 \nabla(g_1)(P_0) + \lambda_2 \nabla(g_2)(P_0) = \vec{0} \end{cases}$$

CT 337.

$$\begin{cases} g_1 = x^2 + y^2 + z^2 - 1 = 0 \\ g_2 = x + y + z = 0 \end{cases}$$

の下で

$$w = f = x + 2y + 3z$$



$$x^2 + y^2 + z^2 = 1$$

$$x + y + z = 0$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \vec{0}$$ を満たす λ_1, λ_2 が存在

$$\begin{pmatrix} 1 & x & 1 \\ 2 & y & 1 \\ 3 & z & 1 \end{pmatrix} \boxed{\begin{pmatrix} 1 \\ 2\lambda_1 \\ \lambda_2 \end{pmatrix}} = \vec{0}$$ となる

$$\neq \vec{0}$$

$$\begin{vmatrix} 1 & x & 1 \\ 2 & y & 1 \\ 3 & z & 1 \end{vmatrix} = 0$$ が必要

$$= x - 2y + z = 0$$

$B : 3 \times 3$.　$\exists \vec{v} \neq \vec{0}$ に対して $B\vec{v} = \vec{0}$

$$\Longleftrightarrow \quad |B| = 0$$

$$
\begin{array}{rl}
 & x + y + z = 0 \\
-) & x - 2y + z = 0 \\
\hline
 & 3y \qquad = 0
\end{array}
$$

より $y = 0$, $x = -z$ が導かれる.

$x^2 + y^2 + z^2 = 1$ に代入

$$x = \pm \frac{\sqrt{2}}{2}, \quad y = 0, \quad z = \mp \frac{\sqrt{2}}{2}$$

が従う.

$g_1 = x + y + z = 0$, $g_2 = x + 2y + 8z - 1 = 0$

の下で $w = f = x^2 + y^2 + z^2$.