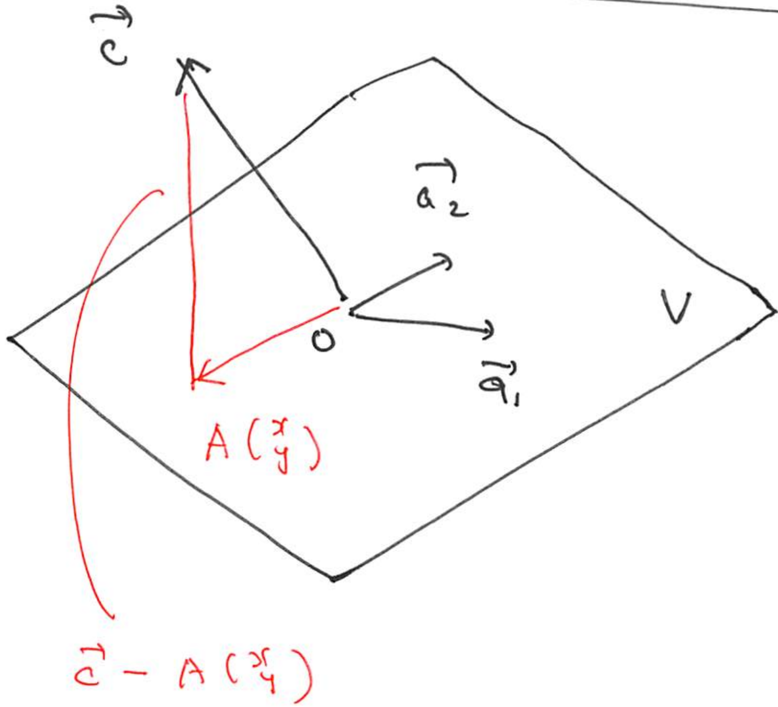
$\vec{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\vec{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$, $\vec{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ $\quad A = (\vec{a}_1\ \vec{a}_2)$ 4行2列

$\| \vec{c} - x\vec{a}_1 - y a_2 \|^2$ を最小にする.

$\| \vec{c} - x\vec{a}_1 - y\vec{a}_2 \|^2$ が最小.



$\iff \vec{c} - A\begin{pmatrix} x \\ y \end{pmatrix} \perp A\begin{pmatrix} p \\ q \end{pmatrix}$

$\left(\begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{R}^2 \right)$

$V = \mathrm{Im}(A) = \left\{ A\begin{pmatrix} x \\ y \end{pmatrix} ;\ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\}$

$\cap$

$\mathbb{R}^4$

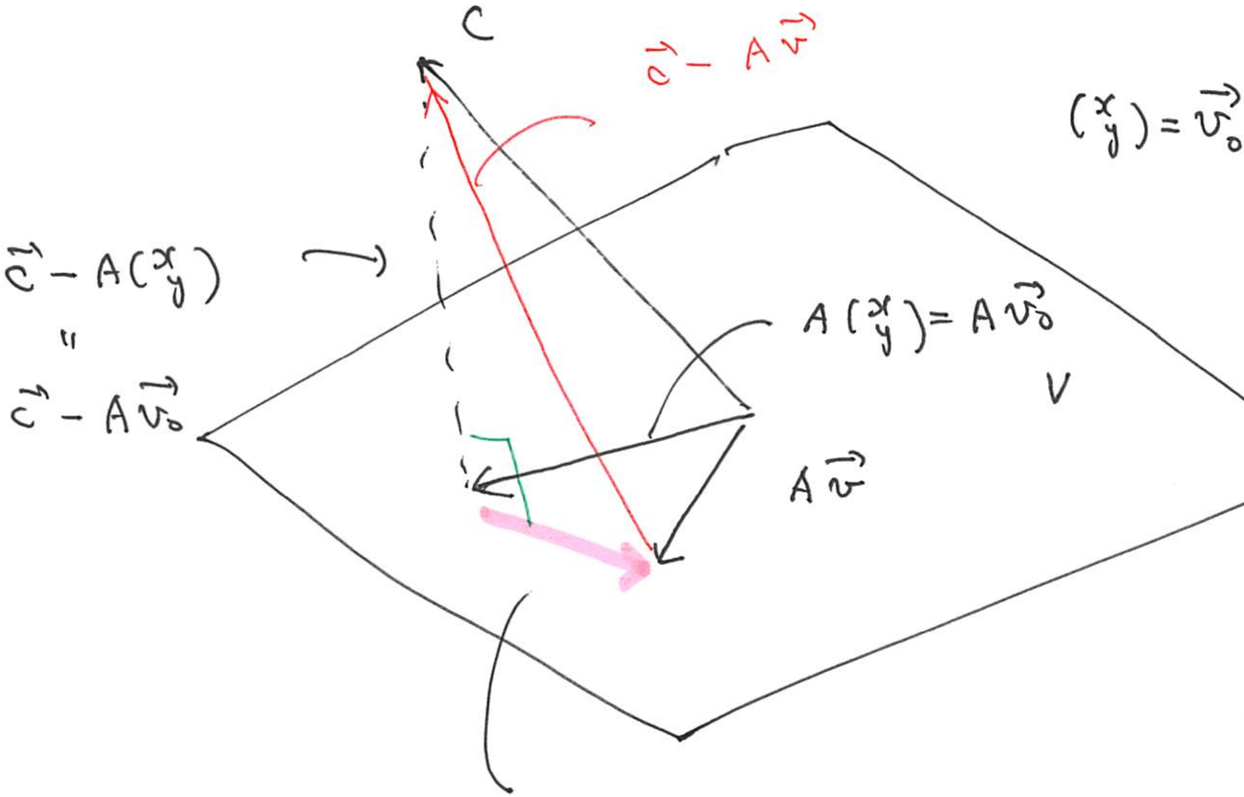
$\iff \left(\vec{c} - A\begin{pmatrix} x \\ y \end{pmatrix},\ A\vec{v} \right) = 0 \quad (\vec{v} \in \mathbb{R}^2)$

$\parallel$

$\left({}^tA\left(\vec{c} - A\begin{pmatrix} x \\ y \end{pmatrix}\right),\ \vec{v} \right)$

ここからわかる.

$\iff \boxed{{}^tA\vec{c} - {}^tAA\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}}$



$$A\vec{v} - A\vec{v}_0 = A(\vec{v} - \vec{v}_0) \in V$$

ピタゴラスの定理. $\vec{\alpha} \perp \vec{\beta} \implies \|\vec{\alpha} \pm \vec{\beta}\|^2 = \|\vec{\alpha}\|^2 + \|\vec{\beta}\|^2$

$$\vec{c} - A\vec{v} = \vec{c} - A\vec{v}_0 + \underbrace{A\vec{v}_0 - A\vec{v}}_{= A(\vec{v}_0 - \vec{v})}$$

$$\boxed{\vec{c} - A\vec{v}_0 \perp A(\vec{v} - \vec{v}_0)}$$

$$\|\vec{c} - A\vec{v}\|^2 = \|\vec{c} - A\vec{v}_0\|^2 + \underbrace{\boxed{\|A\vec{v}_0 - A\vec{v}\|^2}}_{\geq 0} \geq \|\vec{c} - A\vec{v}_0\|^2$$

$$ {}^tAA = \begin{pmatrix} {}^t\vec{a}_1 \\ {}^t\vec{a}_2 \end{pmatrix} (\vec{a}_1 \ \vec{a}_2) = \begin{pmatrix} \|\vec{a}_1\|^2 & (\vec{a}_1, \vec{a}_2) \\ (\vec{a}_2, \vec{a}_1) & \|\vec{a}_2\|^2 \end{pmatrix} $$

Aのグラム行列より

$$ = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} $$

$$ {}^tA\vec{c} = \begin{pmatrix} {}^t\vec{a}_1 \\ {}^t\vec{a}_2 \end{pmatrix} \vec{c} = \begin{pmatrix} (\vec{a}_1, \vec{c}) \\ (\vec{a}_2, \vec{c}) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} $$

$\begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ を解くと $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 3 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

よって $x = \frac{2}{11}$, $y = \frac{3}{11}$.

$$ \boxed{\vec{c} - A\begin{pmatrix} x \\ y \end{pmatrix} \perp V \implies \|\vec{c} - x\vec{a}_1 - y\vec{a}_2\|^2 \text{ は最小.}} $$

$$ \Longleftarrow $$

図2参照, web 図2参照.

"$=$" $\iff \| A\vec{v}_0 - A\vec{v} \| = 0$

$\iff A(\vec{v}_0 - \vec{v}) = \vec{0}. \implies \vec{v}_0 - \vec{v} = \vec{0} \iff \vec{v} = \vec{v}_0$

$\vec{v}_0$ しか最小値をとらない

$\vec{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

$\vec{a}_1 \nparallel \vec{a}_2 \overset{\text{定義}}{\iff} \left(c_1\vec{a}_1 + c_2\vec{a}_2 = \vec{0} \implies c_1 = c_2 = 0 \right)$

$\vec{a}_1 \parallel \vec{a}_2 \iff$ ある $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq \vec{0}$ が存在して $c_1\vec{a}_1 + c_2\vec{a}_2 = \vec{0}$

$\left(A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{0} \implies \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{0} \right)$

n 変量

x_1	x_2	$\cdots$	x_j	$\cdots$	x_n
$x_{1,1}$	$x_{2,1}$	$\cdots$	$x_{j,1}$		
$x_{1,2}$	$x_{2,2}$	$\cdots$	$x_{j,2}$		
$\vdots$	$\vdots$		$\vdots$		
$x_{1,i}$	$x_{2,i}$	$\cdots$	$x_{j,i}$	$\cdots$	$x_{n,i}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$x_{1,N}$	$x_{2,N}$	$\cdots$	$x_{j,N}$	$\cdots$	$x_{n,N}$

$$V(z) = (C\vec{\alpha}, \vec{\alpha})$$

Cは分散行列

V(z)に関係か..

$$\boxed{z = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b.}$$

$$\bar{z} = a_1 \bar{x}_1 + a_2 \bar{x}_2 + \cdots + a_n \bar{x}_n + b$$

$$z_i = a_1 x_{1,i} + a_2 x_{2,i} + \cdots + a_n x_{n,i} + b$$

$$z_i - \bar{z} = a_1 (x_{1,i} - \bar{x}_1) + a_2 (x_{2,i} - \bar{x}_2) + \cdots + a_n (x_{n,i} - \bar{x}_n)$$

$$\boxed{\vec{z} = a_1 \vec{x}_1 + a_2 \vec{x}_2 + \cdots + a_n \vec{x}_n} = \underbrace{\begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{pmatrix}}_{= D} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \vec{\alpha}$$

$$= D\vec{\alpha}$$

$$V(z) = \|\vec{z}\|^2 = (D\vec{\alpha}, D\vec{\alpha}) =$$

$$= ({}^tDD\vec{\alpha}, \vec{\alpha})$$

$$\boxed{(\vec{a}, \vec{b}) = {}^t\vec{a}\,\vec{b}}$$

$${}^tDD = \begin{pmatrix} {}^t\vec{x}_1 \\ {}^t\vec{x}_2 \\ \vdots \\ {}^t\vec{x}_n \end{pmatrix} (\vec{x}_1, \vec{x}_2 \cdots \vec{x}_n)$$

$$= \begin{pmatrix} \|\vec{x}_1\|^2 & (\vec{x}_1, \vec{x}_2) & & (\vec{x}_1, \vec{x}_n) \\ (\vec{x}_2, \vec{x}_1) & \|\vec{x}_2\|^2 & \ddots & (\vec{x}_2, \vec{x}_n) \\ \vdots & \vdots & & \vdots \\ (\vec{x}_n, \vec{x}_1) & (\vec{x}_n, \vec{x}_1) & \cdots & \|\vec{x}_n\|^2 \end{pmatrix}$$

$$= \begin{pmatrix} V(x_1) & Cov(x_1, x_2) & & \\ Cov(x_2, x_1) & V(x_2) & & \\ & & \ddots & \\ & & & V(x_n) \end{pmatrix}$$ 分散行列

対称行列である.

$A : m \times n$, $B : n \times l$. $\longrightarrow$ AB.

m行 n列

I $\quad A = (\vec{a}_1 \cdots \vec{a}_n)$ $\quad \boxed{A \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = b_1 \vec{a}_1 + b_2 \vec{a}_2 + \cdots + b_n \vec{a}_n} \in \mathbb{R}^m$

$\vec{a}_j \in \mathbb{R}^m$

$B := (\vec{b}_1 \cdots \vec{b}_l)$

II $\quad AB = (A\vec{b}_1 \ A\vec{b}_2 \ \cdots \ A\vec{b}_l)$ $\rightsquigarrow$ ABはm行l列.

$A\vec{b}_k \in \mathbb{R}^n$

公式 (転置) $\quad {}^t(AB) = {}^tB\,{}^tA$, $\quad {}^t({}^tA) = A$

$${}^t({}^tDD) = {}^tD\,{}^t({}^tD) = {}^tDD.$$

$\rightsquigarrow$ ${}^tDD$ 対称行列.

$A_1 \quad A_2 \quad \cdots \quad A_n$

$x_1 + x_2 + \cdots + x_n = 1$ として

$r_{1,1}$

A_j に x_j 投資．　R_j：収益率

$r_{1,2}$

$\vdots$

ポートフォリオの収益率

$r_{1,N}$

$$R = x_1 R_1 + x_2 R_2 + \cdots + x_n R_n$$

$\mu_j = E[R_j]$

$\sigma_{ij} = Cov(R_i, R_j)$　　$C = (\sigma_{ij})$ 分散行列

$$V(R) = (C\vec{x}, \vec{x})$$

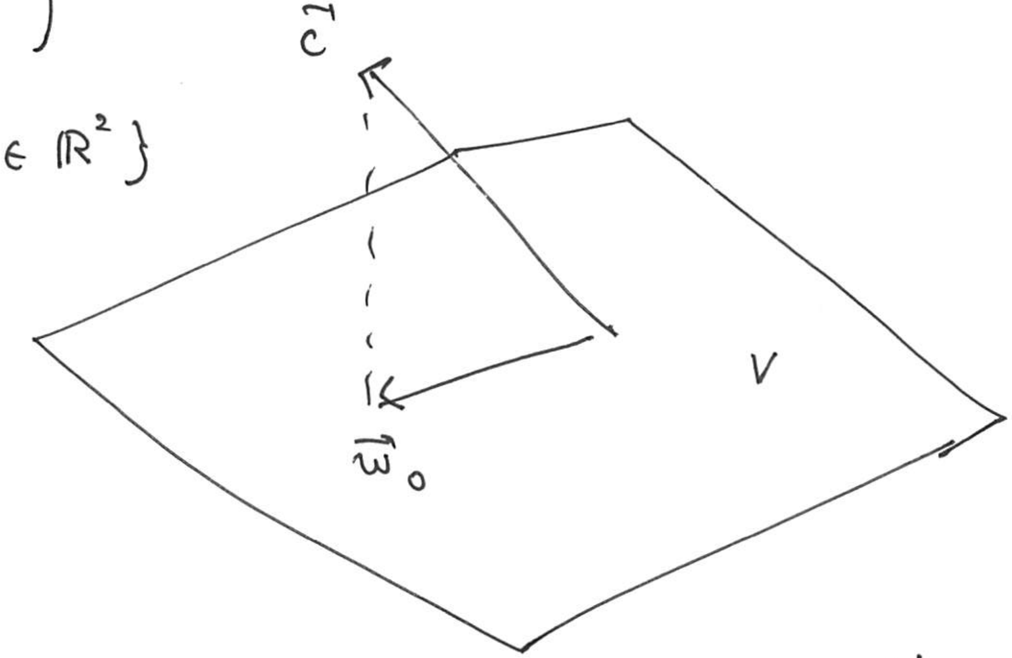
$$\left.\begin{array}{l} \mu = x_1\mu_1 + \cdots + x_n\mu_n \\ 1 = x_1 + \cdots + x_n \end{array}\right)$$

$\mu = E[R]$

の下で　$V(R)$ を最小にする．

2制約条件の下で考える．

$$\vec{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad A = (\vec{a}_1 \ \vec{a}_2)$$



$$V = \mathrm{Im}(A) = \left\{ A \begin{pmatrix} x \\ y \end{pmatrix} ; \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\}$$

$$\begin{cases} \vec{c} - \vec{w}_0 \perp V \\ \vec{w}_0 \in V \end{cases}$$

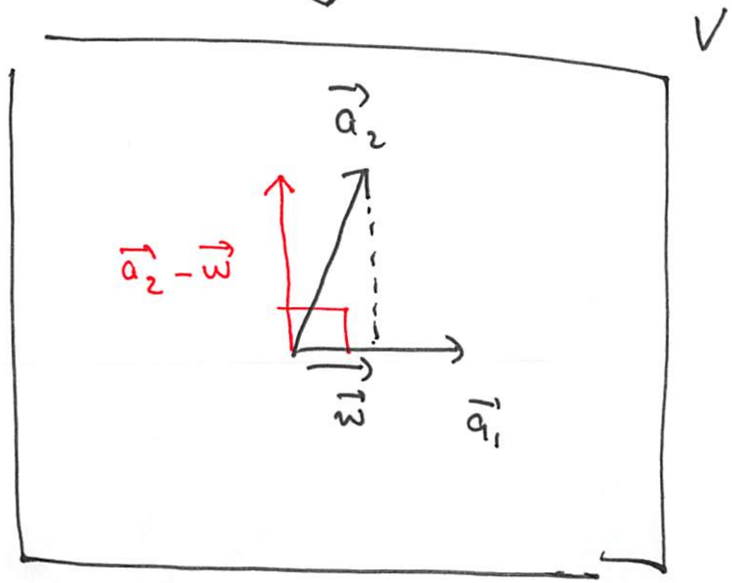
を満たす $\vec{w}_0 \in \mathbb{R}^4$ を求める.

グラム・シュミットの直交化

$\vec{a}_2$ の $\vec{a}_1$ への直交射影 $\vec{w}$

$$\vec{w} = \frac{(\vec{a}_1, \vec{a}_2)}{\|\vec{a}_1\|^2} \vec{a}_1$$

$$= \frac{1}{4} \vec{a}_1$$



$$\vec{a}_2 - \vec{w} = \begin{pmatrix}1\\0\\1\\1\end{pmatrix} - \frac{1}{4}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix} = \begin{pmatrix}\frac{3}{4}\\-\frac{1}{4}\\\frac{3}{4}\\\frac{5}{4}\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3\\-1\\3\\5\end{pmatrix}$$

35

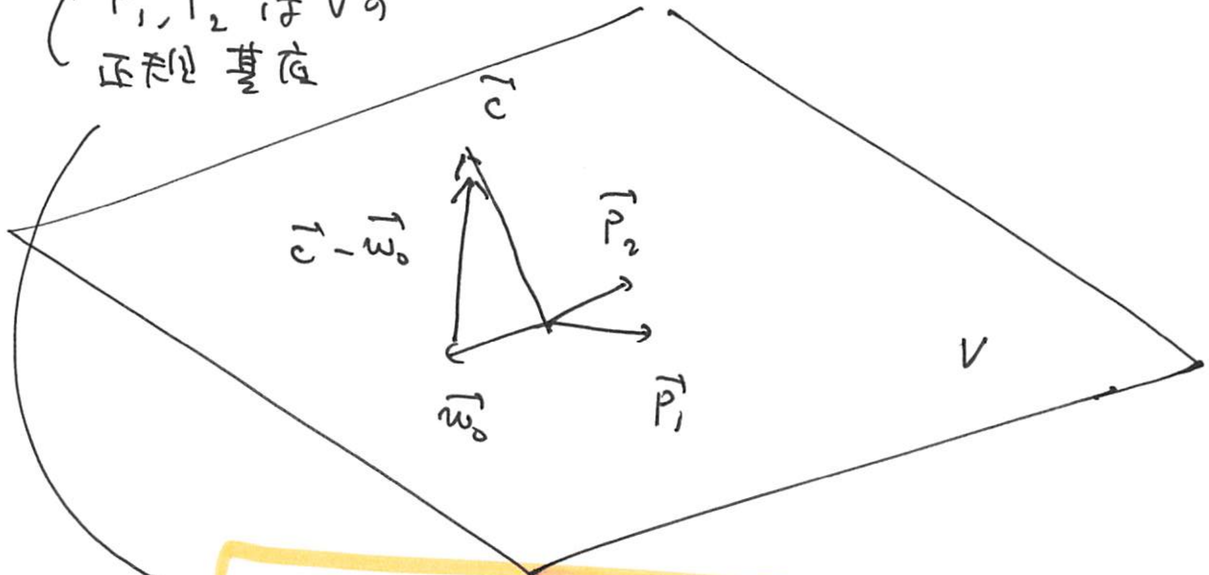
$9+1+9+25$
$= 44.$

$$\vec{p}_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}, \quad \vec{p}_2 = \frac{1}{\sqrt{44}}\begin{pmatrix}3\\-1\\3\\5\end{pmatrix}$$

($\vec{p}_1, \vec{p}_2$ は V の正規基底

$$V \ni \vec{w}_0 = x\vec{p}_1 + y\vec{p}_2$$

と書ける



$$\boxed{\|\vec{p}_1\| = \|\vec{p}_2\| = 1, \quad (\vec{p}_1, \vec{p}_2) = 0}$$

$$\vec{c} - \vec{w}_0 \perp \vec{p}_1, \vec{p}_2$$

$$\Updownarrow$$

$$\vec{c} - \vec{w}_0 \perp V$$

$$(\vec{c} - x\vec{p}_1 - y\vec{p}_2, \underset{2}{\vec{p}_1}) = 0$$

$$\shortparallel$$

$$(\vec{c}, \underset{2}{\vec{p}_1}) - \underset{y}{x} \longrightarrow \underset{y}{x} = (\vec{c}, \underset{\vec{p}_2}{\vec{p}_1}),$$

同様に

$$\vec{w_0} = (\vec{c}, \vec{p_1})\vec{p_1} + (\vec{c}, \vec{p_2})\vec{p_2} = \cdots$$

計算.

$$\vec{a_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{a_2} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A = (\vec{a_1} \ \vec{a_2})$$

$\vec{c}$ の $V = \mathrm{Im}\, A$ への直交射影をグラム・シュミットを用いて求めよ.