emath 2015/10/06

I $\quad \vec{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}$

$$\|\vec{a} \pm \vec{b}\|^2$$
$$= \|\vec{a}\|^2 \pm 2(\vec{a}, \vec{b}) + \|\vec{b}\|^2$$

$$\|\vec{a} - t\vec{b}\|^2$$
$$= \|\vec{a}\|^2 - 2(\vec{a}, t\vec{b}) + \|t\vec{b}\|^2$$
$$= \|\vec{a}\|^2 - 2t(\vec{a}, \vec{b}) + t^2\|\vec{b}\|^2$$

$\|\vec{b}\| > 0.$

$$= \|\vec{b}\|^2 \left(t - \frac{(\vec{a}, \vec{b})}{\|\vec{b}\|^2}\right)^2 + \|\vec{a}\|^2 - \frac{(\vec{a}, \vec{b})^2}{\|\vec{b}\|^2}$$

$$\boxed{\begin{aligned} &\|\vec{a}\|^2 = 1 + 1 + 4 + 1 = 7. \\ &(\vec{a}, \vec{b}) = 2, \quad \|\vec{b}\|^2 = 7 \end{aligned}}$$

$$= 7t^2 - 4t + 7$$
$$= 7\left(t - \frac{2}{7}\right)^2 + 7 - \frac{4}{7}$$

$t = \frac{2}{7}$ のとき $7 - \frac{4}{7} = \frac{45}{7}.$

$$t = \frac{(\vec{a}, \vec{b})}{\|\vec{b}\|^2} = \frac{2}{7}.$$

最小値 $\quad \|\vec{a}\|^2 - \frac{(\vec{a}, \vec{b})^2}{\|\vec{b}\|^2} = 7 - \frac{4}{7} = \frac{45}{7}.$

Ⅰ $\|\vec{b}\| \neq 0 \ (\Longleftrightarrow \vec{b} \neq \vec{0})$ のとき

$$\|\vec{a} - t\vec{b}\|^2 \geqq \frac{\|\vec{a}\|^2 \cdot \|\vec{b}\|^2 - (\vec{a}, \vec{b})^2}{\|\vec{b}\|^2} \geqq 0$$

$$\boxed{\begin{array}{c} \text{\rotatebox{90}{$\leqq$}} \\ 0 \\ (\text{すべての } t \in \mathbb{R}) \end{array}}$$

$$\text{"="} \Longleftrightarrow t = \frac{(\vec{a}, \vec{b})}{\|\vec{b}\|^2}$$

$$\rightsquigarrow \|\vec{a}\|^2 \cdot \|\vec{b}\|^2 \geqq (\vec{a}, \vec{b})^2$$

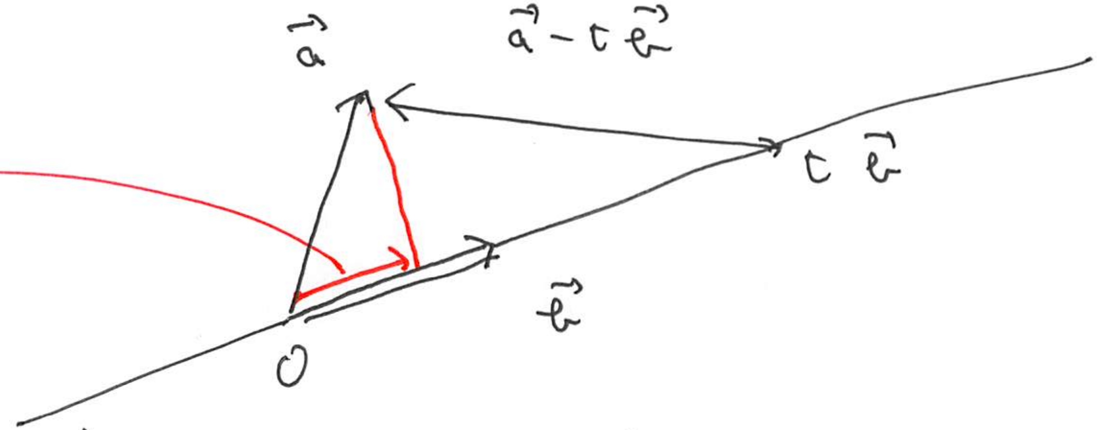
$$\overset{\rightsquigarrow}{\sqrt{\cdot}} \quad |(\vec{a}, \vec{b})| \leqq \|\vec{a}\| \cdot \|\vec{b}\|^2$$

コーシー・シュワルツ.

Ⅱ Not Ⅰ $\vec{b} = \vec{0}$ のとき.

$$\left.\begin{array}{l} (\vec{a}, \vec{b}) = (\vec{a}, \vec{0}) = 0 \\ \|\vec{a}\| \cdot \|\vec{b}\| = \|\vec{a}\| \cdot 0 = 0 \end{array}\right)$$

$$\frac{(\vec{a}, \vec{b})}{\|\vec{b}\|^2}\vec{b}$$



$\|\vec{a} - t\vec{b}\|^2$ が最小 $\iff (\vec{a} - t\vec{b}, \vec{b}) = 0$

$$= (\vec{a}, \vec{b}) - t\|\vec{b}\|^2$$

$$\iff t = \frac{(\vec{a}, \vec{b})}{\|\vec{b}\|^2}$$

$\vec{a}$ の $\vec{b}$ 方向の直交射影

$$\frac{(\vec{a}, \vec{b})}{\|\vec{b}\|^2}\vec{b}.$$

normal

orthogonal 直交.

II $(\vec{a}\ \vec{b}\ \vec{c}\ \vec{d})\begin{pmatrix}1&0&0&\lambda\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = (\vec{a}\ \vec{b}\ \vec{c}\ \ \lambda\vec{a}+\vec{d})$

$$(\vec{a}\ \vec{b}\ \vec{c}\ \vec{d})\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = x\vec{a}+y\vec{b}+z\vec{c}+w\vec{d}$$

重回帰分布.

x	y	z
x_1	y_1	z_1
x_2	y_2	z_2
$\vdots$	$\vdots$	$\vdots$
x_N	y_N	z_N

説明変数　　目的変数.

Explanatory Variable　　Objective Variable.

$$z = ax + by + c$$

$$\boxed{\varepsilon = z - (ax + by + c)}$$

I　$\bar{\varepsilon} = 0$

II　$V(\varepsilon)$ 最小.

最小自乗法

Least Square.

The Method of

I $\longrightarrow$ $\bar{\varepsilon} = \bar{z} - a\bar{x} - b\bar{y} - c = 0$

$$\vec{\varepsilon} = \frac{1}{\sqrt{N}} \begin{pmatrix} \varepsilon_i - \bar{\varepsilon} \end{pmatrix}$$

$$\varepsilon_i = z_i - (ax_i + by_i + c)$$

$$\varepsilon_i - \bar{\varepsilon} = \{z_i - (ax_i + by_i + c)\} - (\bar{z} - a\bar{x} - b\bar{y} - c)$$

$$= (z_i - \bar{z}) - a(x_i - \bar{x}) - b(y_i - \bar{y})$$

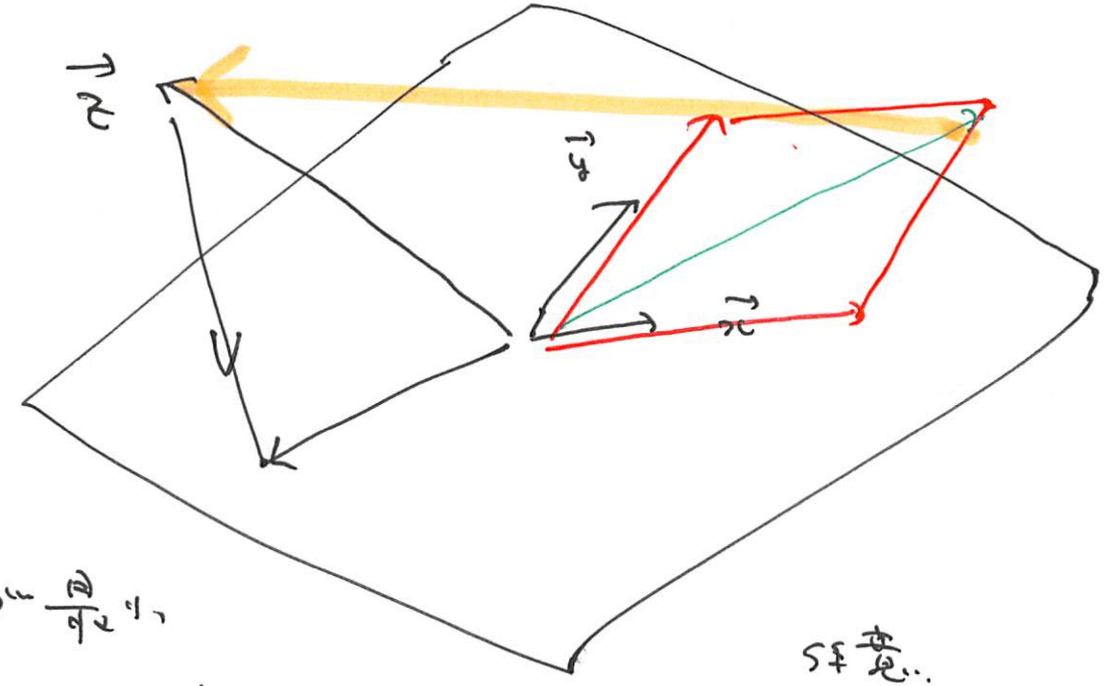
$$\boxed{\vec{\varepsilon} = \vec{z} - a\vec{x} - b\vec{y}}$$

$$\vec{z} = \frac{1}{\sqrt{N}} \begin{pmatrix} \vdots \\ z_i - \bar{z} \\ \vdots \end{pmatrix}$$

$$V(\varepsilon) = \|\vec{\varepsilon}\|^2$$

$\vec{x} \not\parallel \vec{y}$ とする.

$V = \{ a\vec{x} + b\vec{y} \in \mathbb{R}^N ;$
$a, b \in \mathbb{R} \}$



$\| \vec{z} - a\vec{x} - b\vec{y} \|$ が最小より,

既述 $\Leftrightarrow$ $\vec{z} - a\vec{x} - b\vec{y} \perp V$

s,t 任意…

$\Leftrightarrow$ $(\vec{z} - a\vec{x} - b\vec{y}, \ s\vec{x} + t\vec{y}) = 0$ $\left(\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbb{R}^2 \right)$

$D = (\vec{x} \ \vec{y})$ N 行 2 列.

$= \left(\vec{z} - D\begin{pmatrix} a \\ b \end{pmatrix}, \ D\begin{pmatrix} s \\ t \end{pmatrix} \right)$

$= \left({}^t D \left(\vec{z} - D\begin{pmatrix} a \\ b \end{pmatrix} \right), \ \begin{pmatrix} s \\ t \end{pmatrix} \right)$

$\Leftrightarrow$ ${}^t D \vec{z} - {}^t D D \begin{pmatrix} a \\ b \end{pmatrix} = 0$

正規方程式.

$\vec{a} \in \mathbb{R}^N$

$(\vec{a}, \vec{v}) = 0$
$(\forall \vec{v} \in \mathbb{R}^N)$

$\Leftrightarrow$ $\vec{a} = \vec{0}$.

$\Rightarrow$ $\vec{v} = \vec{a}$ にする

$\| \vec{a} \|^2 = (\vec{a}, \vec{a}) = 0$

$${}^tDD \text{ が正則} \iff \vec{x} \nparallel \vec{y}$$
$$D = (\vec{x}\ \vec{y})$$

← 後述

${}^tDD$ は正則。 $\begin{pmatrix} a \\ b \end{pmatrix} = ({}^tDD)^{-1}\, {}^tD\vec{z}$.

$${}^tDD = \begin{pmatrix} {}^t\vec{x} \\ {}^t\vec{y} \end{pmatrix} (\vec{x}\ \vec{y})$$

Dのグラム行列。

$$= \begin{pmatrix} {}^t\vec{x}\,\vec{x} & {}^t\vec{x}\,\vec{y} \\ {}^t\vec{y}\,\vec{x} & {}^t\vec{y}\,\vec{y} \end{pmatrix}$$

$$= \begin{pmatrix} \|\vec{x}\|^2 & (\vec{x}, \vec{y}) \\ (\vec{y}, \vec{x}) & \|\vec{y}\|^2 \end{pmatrix} = \begin{pmatrix} V(x) & C_{xy} \\ C_{yx} & V(y) \end{pmatrix}$$

分散共分散行列。

$$A = (\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n) = \begin{pmatrix} a1_1 \\ a1_2 \\ \vdots \\ a1_m \end{pmatrix}$$

$m \times n$ 行列

$${}^tA = \begin{pmatrix} {}^t\vec{a}_1 \\ {}^t\vec{a}_2 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix} = ({}^ta1_1\ {}^ta1_2 \cdots {}^ta1_m)$$

n行 m列。

$$(\vec{a}, \vec{b}) = {}^t\vec{a}\ \vec{b}$$
$$\vec{a}, \vec{b} \in \mathbb{R}^N$$

$$= (a_1\, a_2 \cdots a_N) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}$$

$${}^t D \vec{z} = \begin{pmatrix} {}^t\vec{x} \\ {}^t\vec{y} \end{pmatrix} \vec{z} = \begin{pmatrix} {}^t\vec{x}\,\vec{z} \\ {}^t\vec{y}\,\vec{z} \end{pmatrix} = \begin{pmatrix} (\vec{x}, \vec{z}) \\ (\vec{y}, \vec{z}) \end{pmatrix}$$

$$= \begin{pmatrix} C_{xz} \\ C_{yz} \end{pmatrix}$$

$$\boxed{\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} V(x) & C_{xy} \\ C_{yx} & V(y) \end{pmatrix}^{-1} \begin{pmatrix} C_{xz} \\ C_{yz} \end{pmatrix} \\ c &= \bar{z} - a\bar{x} - b\bar{y}. \end{aligned}}$$

A : $m \times n$ 行列. $\vec{v} \in \mathbb{R}^n \rightsquigarrow A\vec{v} = v_1 \vec{a}_1 + \cdots + v_n \vec{a}_n \in \mathbb{R}^{m}$

$\parallel$ $(\vec{a}_1 \cdots \vec{a}_n)$

$\vec{a}_j \in \mathbb{R}^{m}$ $\qquad \vec{w} \in \mathbb{R}^{m}$

$$(\underset{\underset{\mathbb{R}^m}{\cap}}{A\vec{v}}, \vec{w}) = (\vec{v}, {}^tA\vec{w})$$

$$
\begin{aligned}
{}^t\mathrm{I} &= (v_1\vec{a}_1 + v_2\vec{a}_2 + \cdots + v_n\vec{a}_n, \vec{w}) \\
&= v_1(\vec{a}_1, \vec{w}) + v_2(\vec{a}_2, \vec{w}) + \cdots + v_n(\vec{a}_n, \vec{w}) \\
&= v_1\,{}^t\vec{a}_1\vec{w} + v_2\,{}^t\vec{a}_2\vec{w} + \\
&\qquad \cdots + v_n\,{}^t a_n\vec{w}) \\
&= \left(\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \boxed{\begin{pmatrix} {}^t\vec{a}_1\vec{w} \\ {}^t\vec{a}_2\vec{w} \\ \vdots \\ {}^t\vec{a}_n\vec{w} \end{pmatrix}} \right)
\end{aligned}
$$

$$
\boxed{\begin{aligned} &(\vec{a}, \vec{b}) \\ &= {}^t\vec{a}\,\vec{b} \end{aligned}}
$$

$$
= \begin{pmatrix} {}^t\vec{a}_1 \\ {}^t\vec{a}_2 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix}\vec{w} = {}^tA\vec{w}
$$

$$\begin{pmatrix} \mathbb{b}_1 \\ \mathbb{b}_2 \\ \vdots \\ \mathbb{b}_n \end{pmatrix} \vec{w} = \begin{pmatrix} \mathbb{b}_1 \vec{w} \\ \mathbb{b}_2 \vec{w} \\ \vdots \\ \mathbb{b}_n \vec{w} \end{pmatrix}$$

n 行 m 列

$\mathbb{b}_j$: m 次元行ベクトル

$\vec{w} \in \mathbb{R}^m$

$$\boxed{(a_1 \cdots a_n) \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = a_1 e_1 + \cdots + a_n e_n}$$

問 $\vec{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\vec{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$, $\vec{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ $\quad A = (\vec{a}_1\ \vec{a}_2)$

$\boxed{\|\vec{c} - x\vec{a}_1 - y\vec{a}_2\|^2 \text{ を最小にする}}$ x, y を求める.

ただし ${}^tAA$ を用いる.

$\Leftrightarrow$ $\quad \left(\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbb{R}^2\right)$

$\left(\vec{c} - A\begin{pmatrix} x \\ y \end{pmatrix}, A\begin{pmatrix} s \\ t \end{pmatrix}\right) = 0$