(1) $\displaystyle\int_1^3 \frac{1}{(2t+1)^3}\,dt = \frac{1}{2}\int_1^3 \frac{(2t+1)'}{(2t+1)^3}\,dt = \frac{1}{2}\int_3^7 \frac{dx}{x^3} = \frac{1}{2}\left[-\frac{1}{2}\cdot\frac{1}{x^2}\right]_3^7$

$x = 2t+1$ とおくと

$$\begin{array}{c|c} t & 1 \nearrow 3 \\ \hline x & 3 \nearrow 7 \end{array}$$

$\left(\frac{1}{x^2}\right)' = -\frac{2}{x^3}$ より $\left(-\frac{1}{2}\frac{1}{x^2}\right)' = \frac{1}{x^3}$

$$= -\frac{1}{4}\left[\frac{1}{7^2} - \frac{1}{3^2}\right] = \frac{10}{441}$$

(別解)

$$\left\{\frac{1}{(2t+1)^2}\right\}' = -\frac{2}{(2t+1)^3}\cdot 2 \text{ より}$$

$$\left(-\frac{1}{4}\cdot\frac{1}{(2t+1)^2}\right)' = \frac{1}{(2t+1)^3}$$

であるので

$$(\text{与式}) = \left[-\frac{1}{4}\cdot\frac{1}{(2t+1)^2}\right]_1^3 = \cdots$$

(2) $\displaystyle\int_0^1 \frac{x-3}{(2-x)^3}\,dx = \int_2^1 \frac{-1-t}{t^3}\,(-dt)$

$t = 2 - x.$

$dt = -dx$

$$\begin{array}{c|ccc} x & 0 & \nearrow & 1 \\ \hline t & 2 & \searrow & 1 \end{array}$$

$x = 2 - t$

$$= -\int_2^1 \left(\frac{1}{t^3} + \frac{1}{t^2}\right) dt$$

$$= -\left[-\frac{1}{2}\cdot\frac{1}{t^2} - \frac{1}{t}\right]_1^2 = -\frac{7}{8}$$

$\left(\frac{1}{t^2}\right)' = -\frac{2}{t^3} \rightsquigarrow \left(-\frac{1}{2}\cdot\frac{1}{t^2}\right)' = \frac{1}{t^3}$

$\left(\frac{1}{t}\right)' = -\frac{1}{t^2} \rightsquigarrow \left(-\frac{1}{t}\right)' = \frac{1}{t^2}$

(3) $\displaystyle\int_1^2 x\sqrt{2-x}\,dx = \int_1^0 (2-t)\sqrt{t}\,(-dt) = \int_0^1 (2-t)\sqrt{t}\,dt$

$t = 2-x$

$dt = -dx$

$x = 2-t$

x	$1 \nearrow 2$
t	$1 \searrow 0$

$\displaystyle = \left[\frac{4}{3}t\sqrt{t} - \frac{2}{5}t^2\sqrt{t}\right]_0^1$

$\displaystyle = \frac{4}{3} - \frac{2}{5} = \frac{14}{15}$

$\displaystyle (t\sqrt{t})' = \frac{3}{2}\sqrt{t} \rightsquigarrow \left(\frac{2}{3}t\sqrt{t}\right)' = \sqrt{t}$

$\displaystyle (t^2\sqrt{t})' = \frac{5}{2}t\sqrt{t} \rightsquigarrow \left(\frac{2}{5}t^2\sqrt{t}\right)' = t\sqrt{t}$

$t = \sqrt{2-x}. \longrightarrow t^2 = 2-x \rightsquigarrow 2t\,dt = -dx$

$x = 2-t^2$

x	$1 \nearrow 2$
t	$1 \searrow 0$

$\displaystyle (\text{与式}) = \int_1^0 (2-t^2)\,t\,\underbrace{(-2t\,dt)}_{=dx}$

$\displaystyle = 2\int_0^1 t^2(2-t^2)\,dt = \cdots$

(4) $\displaystyle\int_0^1 t\sqrt{1+t^2}\,dt = \int_1^2 \sqrt{x}\,\frac{1}{2}\,dx = \frac{1}{2}\left[\frac{2}{3}x\sqrt{x}\right]_1^2$

$x = 1+t^2$

$dx = 2t\,dt$

$$\begin{array}{c|c} t & 0 \nearrow 1 \\ \hline x & 1 \nearrow 2 \end{array}$$

$(x\sqrt{x})' = \frac{3}{2}\sqrt{x}$

$= \frac{1}{3}(2\sqrt{2}-1)$

$\left\{(1+t^2)^{\frac{3}{2}}\right\}' = \frac{3}{2}(1+t^2)^{\frac{1}{2}}\cdot 2t = 3(1+t^2)^{\frac{1}{2}}\cdot t$

$\left\{\frac{1}{3}(1+t^2)^{\frac{3}{2}}\right\}' = (1+t^2)^{\frac{1}{2}}\cdot t$

(5)

$$\int_0^1 t \log(1+t^2)\,dt = \int_1^2 \log x \,\frac{1}{2}\,dx = \frac{1}{2}\int_1^2 (x)' \log x \, dx$$

$x = 1+t^2 \qquad dx = 2t\,dt$

$$= \frac{1}{2}\left[x \log x\right]_1^2 - \frac{1}{2}\int_1^2 \underbrace{x \cdot \frac{1}{x}}_{=1}\, dx$$

$$\boxed{(\log x)' = \frac{1}{x}}$$

$\log 1 = 0$

$$= \frac{1}{2}\cdot 2\log 2 - \frac{1}{2} = \log 2 - \frac{1}{2}$$

(6) $\displaystyle\int_0^1 \frac{e^t\,dt}{(1+e^t)^2} = \int_2^{1+e} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_2^{1+e}$

$x = 1+e^t$

$dx = e^t \cdot dt$

$\begin{array}{c|c} t & 0 \nearrow 1 \\ \hline x & 2 \nearrow 1+e \end{array}$

$= -\left(\frac{1}{1+e} - \frac{1}{2}\right)$

$\boxed{(e^t)' = e^t}$

$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$= \dfrac{e-1}{2(1+e)}$

(7) $\int_0^1 t e^{-\frac{1}{2}t^2}\,dt = \int_0^{-\frac{1}{2}} e^{x}(-dx)$

$\left(-\frac{1}{2}t^2\right)' = -t$

$x = -\frac{1}{2}t^2$ とすると $dx = -t\,dt$

t	$0 \to 1$
x	$0 \to -\frac{1}{2}$

$= \int_{-\frac{1}{2}}^{0} e^{x}\,dx = \left[e^{x}\right]_{-\frac{1}{2}}^{0}$

$= 1 - \frac{1}{\sqrt{e}} = \frac{\sqrt{e}-1}{\sqrt{e}}$

$\left(e^{-\frac{1}{2}t^2}\right)' = -t\,e^{-\frac{1}{2}t^2}$

$\left(-e^{-\frac{1}{2}t^2}\right)' = t\,e^{-\frac{1}{2}t^2}$

$= \left[-e^{-\frac{1}{2}t^2}\right]_0^1 = \cdots$

(8)

$$\int_{-1}^{1} (x+1)^3 (x-1)\,dx = \int_{-1}^{1} \left(\frac{1}{4}(x+1)^4\right)' (x-1)\,dx$$

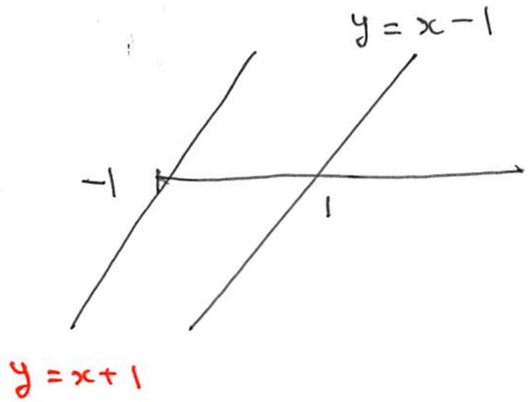
$$\left\{\frac{1}{4}(x+1)^4\right\}' = (x+1)^3$$

$$= \underbrace{\frac{1}{4}\Big[(x+1)^4 (x-1)\Big]_{-1}^{1}}_{=0} - \frac{1}{4}\int_{-1}^{1} (x+1)^4\,dx$$

$$\left((x+1)^5\right)' = 5(x+1)^4$$

$$= -\frac{1}{4}\left[\frac{1}{5}(x+1)^5\right]_{-1}^{1} = -\frac{1}{20}\cdot 2^5$$

$$= -\frac{8}{5}$$



(9) $\displaystyle\int_{-1}^{1} (x+1)^3 (x-1)^2 \, dx$

$$= \int_{-1}^{1} \left(\frac{1}{4}(x+1)^4 \right)' (x-1)^2 \, dx$$

$$\left((x-1)^2 \right)' = 2(x-1)$$

$$= \frac{1}{4} \Big[\overbrace{(x+1)^4 (x-1)^2}^{0 \, ||} \Big]_{-1}^{1} - \frac{1}{4}\int_{-1}^{1} (x+1)^4 \cdot 2(x-1) \, dx$$

$$= -\frac{1}{2} \int_{-1}^{1} \left(\frac{1}{5}(x+1)^5 \right)' (x-1) \, dx$$

$$(x-1)' = 1$$

$$= \Big[\frac{1}{5} \underbrace{(x+1)^5 (x-1)}_{|| \, 0} \Big]_{-1}^{1} + \frac{1}{10} \int_{-1}^{1} (x+1)^5 \, dx$$

$$= \frac{1}{10} \left[\frac{1}{6}(x+1)^6 \right]_{-1}^{1} = \frac{2^6}{60} = \frac{16}{15}$$