I (1) $\int_0^1 (1+t^2)^4 t\,dt = \frac{1}{2}\int_0^1 (1+t^2)^4 (1+t^2)'\,dt = \frac{1}{2}\int_1^2 x^4\,dx$

$$\boxed{\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \int_A^B f(x)\,dx}$$

$A = \varphi(a)$, $B = \varphi(b)$

$x = \varphi(t) = 1+t^2$

t	$0 \to 1$
x	$1 \to 2$

$(1+t^2)' = 2t$

$= \frac{1}{2}\left[\frac{1}{5}x^5\right]_1^2 = \frac{1}{10}(2^5-1) = \frac{31}{10}$

$\{(1+t^2)^5\}' = 5(1+t^2)^4 \cdot 2t$

$\left\{\frac{1}{10}(1+t^2)^5\right\}' = (1+t^2)^4 \cdot t$

$= \frac{1}{10}\int_0^1 (1+t^2)^5\,dt = \frac{1}{10}\left[(1+t^2)^5\right]_0^1$

$= \cdots$

(2) $\displaystyle\int_0^1 (1-t)^6\,dt = -\int_0^1 (1-t)^6 \underbrace{(-1)}_{=\varphi'(t)}\,dt =$

$(1-t)' = -1$ $\qquad\qquad x = \varphi(t) = 1-t$

t	$0 \to 1$
x	$1 \to 0$

$\displaystyle = -\int_0^1 (1-t)^6 (1-t)'\,dt$

$\displaystyle = \underbrace{\boxed{-\int_1^0}}_{=\int_0^1} x^6\,dx = \left[\frac{1}{7}x^7\right]_0^1 = \frac{1}{7}$

$((1-t)^7)' = 7(1-t)^6 \cdot (-1)$

(与式) $\displaystyle = -\frac{1}{7}\int_0^1 \{(1-t)^7\}'\,dt = -\frac{1}{7}\left[(1-t)^7\right]_0^1 = \cdots$

$\displaystyle\int_0^1 (1-x)^6\,dx = \int_1^0 t^6 (-1)\,dt$

t	$1 \to 0$
x	$0 \to 1$

$1-x=t$ とおけば $x=\varphi(t)=1-t$, このとき $\varphi'(t)=-1$

(3) $\displaystyle\int_0^1 t(1-t)^5\,dt =$

$$\int_a^b f' g = [fg]_a^b - \int_a^b f g'$$

$$\{(1-t)^6\}' = 6(1-t)^5\cdot(-1)$$

$$-\frac{1}{6}\{(1-t)^6\}' = (1-t)^5$$

$$= -\frac{1}{6}\int_0^1 t\,\{(1-t)^6\}'\,dt$$

$$= -\frac{1}{6}\left[t(1-t)^6\right]_0^1 + \frac{1}{6}\int_0^1 (1-t)^6\,dt$$

$$= 0 + \frac{1}{6}\cdot\frac{1}{7} = \frac{1}{42}$$

(2)

$$I = \int_0^1 x(1-x)^5\,dx = \int_1^0 (1-t)\cdot t^5\,(-1)\,dt = \int_0^1 (t^5 - t^6)\,dt$$

$x = \varphi(t) = 1-t$ とおくと $\varphi'(t) = (-1)$

$1-x = t \rightsquigarrow x = 1-t$

t	$1 \to 0$
x	$0 \to 1$

$$= \left[\frac{t^6}{6} - \frac{t^7}{7}\right]_0^1 = \frac{1}{6} - \frac{1}{7} = \cdots$$

(4) $\displaystyle\int_0^1 (2x+1)^4\,dx = \int_1^3 t^4 \cdot \frac{1}{2}\,dt$

t	$1 \nearrow 3$
x	$0 \nearrow 1$

$t = 2x+1$ とするように $x = \varphi(t) = \dfrac{t-1}{2}$ とおくと $\varphi'(t) = \dfrac{1}{2}$

$$= \frac{1}{2}\left[\frac{t^5}{5}\right]_1^3 = \frac{1}{10}(3^5 - 1) = \frac{242}{10}$$

$$= \frac{121}{5}$$

$$\begin{array}{r} 81 \\ \times\ 3 \\ \hline 243 \end{array}$$

$\{(2x+1)^5\}' = 5(2x+1)^4 \cdot 2$

$$= \frac{1}{10}\int_0^1 \{(2x+1)^5\}' = \frac{1}{10}\left[(2x+1)^5\right]_0^1 = \cdots$$

(5)

$$\int_0^1 \frac{t}{(1+t^2)^2}\,dt = \frac{1}{2}\int_0^1 \frac{\boxed{(1+t^2)'}}{(1+t^2)^2}\,dt \qquad = \varphi'(t)$$

t	$0 \nearrow 1$
x	$1 \nearrow 2$

$(1+t^2)' = 2t$

$$= \frac{1}{2}\int_1^2 \frac{1}{x^2}\,dx.$$

$x = \varphi(t) = 1+t^2$

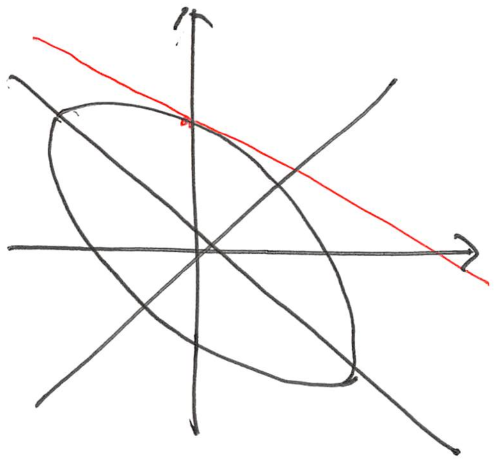
$$= \frac{1}{2}\left[-\frac{1}{x}\right]_1^2 = \frac{1}{2}\left(-\frac{1}{2}+1\right) = \frac{1}{4}.$$

$\left(-\frac{1}{x}\right)' = \frac{1}{x^2}$

$$g(x,y) = x^2 + xy + y^2 - 1 = 0$$

$$= \left(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

$$\begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = 1 - \frac{1}{4} = \frac{3}{4} > 0$$



$(0,1)$ の近くで　　$y^2 + xy + x^2 - 1 = 0$

$$y = \frac{-x \pm \sqrt{x^2 - 4(x^2-1)}}{2} = \frac{-x \pm \sqrt{4 - 3x^2}}{2}$$

$$\varphi(x) = \frac{-x + \sqrt{4 - 3x^2}}{2}$$

$$\varphi'(0) = -\frac{g_x(0,1)}{g_y(0,1)} = -\frac{1}{2}$$

$P_0(0,1)$

$$g_x = 2x + y, \quad g_y = x + 2y.$$

$$\varphi''(0) = \frac{1}{g_y(0,1)^3} \begin{vmatrix} 0 & g_x(P_0) & g_y(P_0) \\ g_x(P_0) & g_{xx}(\ ) & g_{xy}(\ ) \\ g_y(P_0) & g_{yx}(\ ) & g_{yy}(\ ) \end{vmatrix}$$

$$g_{xx} = 2,\quad g_{xy} = g_{yx} = 1,\quad g_{yy} = 2$$

—— マイナス

—— プラス

$$\varphi''(0) = \frac{1}{2^3}\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix} = \frac{1}{2^3}\left(-2^3 - 2\cdot 1^2 + 2\cdot 2\cdot 1^2\right)$$

$$= -\frac{3}{4}$$

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & -3 & 0 \end{vmatrix} = -1\begin{vmatrix} 1 & 2 \\ -3 & 0 \end{vmatrix} = -6.$$

$$\begin{array}{rrrr} & 2 & 1 & 2 \\ -) & 2 & 4 & 2 \\ \hline & 0 & -3 & 0 \end{array}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$x^2 + x\varphi(x) + \varphi(x)^2 - 1 \equiv 0$$

$$(fg)' = f'g + fg'$$

$$\longrightarrow \quad 2x + 1\cdot\varphi(x) + x\varphi'(x) + 2\varphi(x)\varphi'(x) \equiv 0$$

$$\boxed{\varphi'(x) = -\frac{2x + \varphi(x)}{x + 2\varphi(x)}}$$

$$\varphi'(0) = -\frac{1}{2} \qquad \varphi(0) = 1$$

$$\longrightarrow \quad 2 + \varphi'(x) + 1\cdot\varphi'(x) + x\cdot\varphi''(x)$$

$$+ 2\varphi'(x)\cdot\varphi'(x)$$

$$+ 2\varphi(x)\varphi''(x) \equiv 0$$

$$2 + 2\varphi' + 2(\varphi')^2 + x\varphi'' + 2\varphi\varphi'' \equiv 0$$

$$\varphi'' = -\frac{2 + 2\varphi'(x) + 2(\varphi'(x))^2}{x + 2\varphi(x)}$$

確率分布　　　$f : \mathbb{R} \to \mathbb{R}$

(i) $f(x) \geq 0$

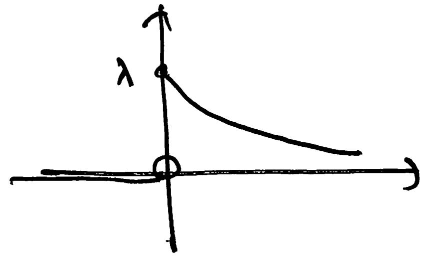
(ii) $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = 1$　　全確率 1

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

$a \leq X \leq b$ となる確率

<u>指数分布</u>　　$\lambda > 0$ 固定

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & (x \geq 0) \\ 0 & (x < 0) \end{cases}$$

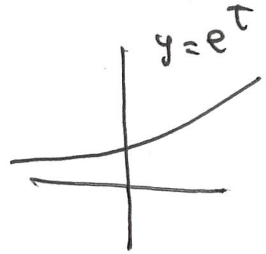


$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty} f(x)\,dx$$

$$= \lim_{R \to +\infty} \int_0^R \lambda e^{-\lambda x}\,dx = 1$$

$$(e^{-\lambda x})' = -\lambda e^{-\lambda x}$$

$$\int_0^R \lambda e^{-\lambda x} = -\int_0^R (e^{-\lambda x})' \, dx$$



$$= -[e^{-\lambda x}]_0^R = -(e^{-\lambda R} - 1)$$

$$= 1 - e^{-\lambda R}$$

$-\lambda R \to -\infty$

$$\to 1 - 0 = 1$$

$$E[x] = \int_0^{+\infty} x f(x) \, dx = \int_0^{+\infty} x \lambda e^{-\lambda x} \, dx$$

$$= \lim_{x \to +\infty} \int_0^R x \lambda e^{-\lambda x} \, dx = \frac{1}{\lambda}$$

$$\int_0^R x \lambda e^{-\lambda x} \, dx = \int_0^R x (-e^{-\lambda x})' \, dx \qquad (e^{-\lambda x})' = -\lambda e^{-\lambda x}.$$

$$= -[x e^{-\lambda x}]_0^R + \int_0^R 1 \cdot e^{-\lambda x} \, dx$$

$$= -R e^{-\lambda R} + \int_0^R \left(-\frac{1}{\lambda} e^{-\lambda x}\right)' dx$$

$$= -Re^{-\lambda R} + \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^R$$

$-\lambda R \to -\infty$

$$= -R\underbrace{e^{-\lambda R}}_{\to 0} - \frac{1}{\lambda}\underbrace{e^{-\lambda R}}_{\to 0} + \frac{1}{\lambda} \to \frac{1}{\lambda}$$

$$\frac{R}{e^{\lambda R}} = \frac{1}{\lambda}\cdot\frac{\lambda R}{e^{\lambda R}} \to \frac{1}{\lambda}0 = 0$$

$$\boxed{\frac{t^4}{e^t} \to 0 \quad (t \to +\infty)}$$

例

$$E[X^2] = \int_0^{+\infty} x^2 \lambda e^{-\lambda x}\,dx = \cdots = \frac{2}{\lambda^2}$$

↑ 2回.

$$V[X] = E[X^2] - (E[X])^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

補足

$$E[X^2] = \int_0^{+\infty} x^2 \lambda e^{-\lambda x} dx = \lim_{R \to +\infty} \int_0^R x^2 \lambda e^{-\lambda x} dx$$

$$\int_0^R x^2 \lambda e^{-\lambda x} dx$$

$$= -\int_0^R x^2 (e^{-\lambda x})' dx$$

$$= -[x^2 e^{-\lambda x}]_0^R + \int_0^R 2x e^{-\lambda x} dx$$

$$= -R^2 e^{-\lambda R} + 2\int_0^R x \left(-\frac{1}{\lambda} e^{-\lambda x}\right)' dx$$

$$= -R^2 e^{-\lambda R} - \frac{2}{\lambda}[x e^{-\lambda x}]_0^R + \frac{2}{\lambda}\int_0^R e^{-\lambda x} dx$$

$$= -R^2 e^{-\lambda R} - \frac{2}{\lambda} R e^{-\lambda R} + \frac{2}{\lambda}\left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^R$$

$$= -R^2 e^{-\lambda R} - \frac{2}{\lambda} R e^{-\lambda R} - \frac{2}{\lambda^2} e^{-\lambda R} + \frac{2}{\lambda^2}$$

$R \to +\infty$ のとき $\dfrac{R^2}{e^{\lambda R}} = \dfrac{1}{\lambda^2} \cdot \dfrac{(\lambda R)^2}{e^{\lambda R}} \to 0$, $\dfrac{R}{e^{\lambda R}} = \dfrac{1}{\lambda} \cdot \dfrac{\lambda R}{e^{\lambda R}} \to 0$

$e^{-\lambda R} \to 0$ より

$$\int_0^R x^2 \lambda e^{-\lambda x} \, dx \longrightarrow \frac{1}{\lambda^2}.$$

F, 2 $\qquad \mathbb{E}[X^2] = \dfrac{2}{\lambda^2}$

(1) $\displaystyle\int_1^3 \frac{1}{(2t+1)^3}\,dt$

(2) $\displaystyle\int_0^1 \frac{x-3}{(2-x)^3}\,dx$

(3) $\displaystyle\int_1^2 x\sqrt{2-x}\,dx$

$t = 2-x.$

$t = \sqrt{2-x}.$

(4) $\displaystyle\int_0^1 t\sqrt{1+t^2}\,dt$

(5) $\displaystyle\int_0^1 t\log(1+t^2)\,dt$

(6) $\displaystyle\int_0^1 \frac{e^t}{(1+e^t)^2}\,dt$

(7) $\displaystyle\int_0^1 t\,e^{-\frac{1}{2}t^2}\,dt$

(8) $\displaystyle\int_{-1}^1 (x+1)^3(x-1)\,dx$

(9) $\displaystyle\int_{-1}^1 \boxed{(x+1)^3}(x-1)^2\,dx.$

$$\left\{\frac{(x+1)^4}{4}\right\}'$$