Calc NT

2016/01/13

$$\boxed{\int_a^b f'g = [fg]_a^b - \int_a^b fg'}$$

$(e^{2t})' = 2e^{2t}$

$(\frac{1}{2}e^{2t})' = e^{2t}$

(1) $\displaystyle \int_0^1 t\,e^{2t}\,dt = \int_0^1 t\left(\frac{1}{2}e^{2t}\right)'dt$

$\displaystyle = \left[t\cdot\frac{1}{2}e^{2t}\right]_0^1 - \int_0^1 1\cdot\left(\frac{1}{2}e^{2t}\right)dt$

$\displaystyle = \frac{1}{2}e^2 - \frac{1}{2}\left[\frac{1}{2}e^{2t}\right]_0^1 = \frac{1}{2}e^2 - \frac{1}{4}(e^2-1)$

$\displaystyle = \frac{1}{4}e^2 + \frac{1}{4} = \frac{e^2+1}{4}$

$(e^{-2t})' = -2e^{-2t}$

$(-\frac{1}{2}e^{-2t})' = e^{-2t}$

(2) $\displaystyle \int_0^1 t^2 e^{-2t}\,dt = \int_0^1 t^2\left(-\frac{1}{2}e^{-2t}\right)'dt$

$\displaystyle = -\frac{1}{2}\left[t^2 e^{-2t}\right]_0^1 + \frac{1}{2}\int_0^1 2t\; e^{-2t}\,dt$

$\displaystyle = -\frac{1}{2}(e^{-2} - 0) + \int_0^1 t\left(-\frac{1}{2}e^{-2t}\right)'dt$

$$= -\frac{1}{2}e^{-2} - \frac{1}{2}\left[t\, e^{-2t}\right]_0^1 + \frac{1}{2}\int_0^1 1 \cdot e^{-2t}\, dt$$

$$\uparrow \ (t)'$$

$$= -\frac{1}{2}e^{-2} - \frac{1}{2}e^{-2} + \frac{1}{2}\left[-\frac{1}{2}e^{-2t}\right]_0^1$$

$$= -e^{-2} - \frac{1}{4}\left(e^{-2} - 1\right) = -\frac{5}{4}e^{-2} + \frac{1}{4} = \frac{e^2 - 5}{4e^2} > 0.$$

$e > 2.5$

$e^2 > 6.25.$

(3) $$\int_1^{e^2} x \log x \, dx = \int_1^{e^2} \left(\frac{x^2}{2}\right)' \log x \, dx$$

$(\log x)' = \frac{1}{x}$

$$= \left[\frac{x^2}{2}\log x\right]_1^{e^2} - \frac{1}{2}\int_1^{e^2} x^2 \cdot \frac{1}{x}\, dx$$

$$= \frac{e^4}{2}\cdot 2 - \frac{1}{2}\int_1^{e^2} x\, dx = e^4 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^{e^2}$$

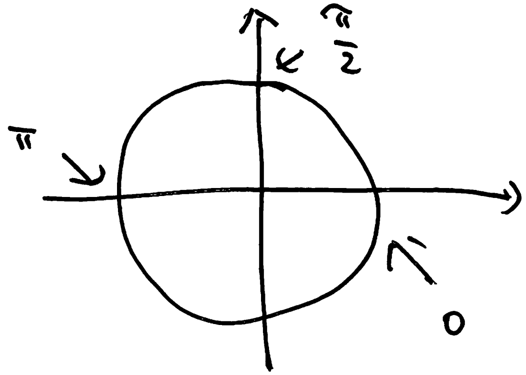
$\uparrow$ $\log e^2$

$$= e^4 - \frac{1}{4}\left(e^4 - 1\right) = \frac{3e^4 + 1}{4}$$

(4) $\int_0^{\frac{\pi}{2}} \sin x \, dx = \left[-\cos x\right]_0^{\frac{\pi}{2}}$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$



$= -(0-1) = 1$

$(-\cos x)' = \sin x$

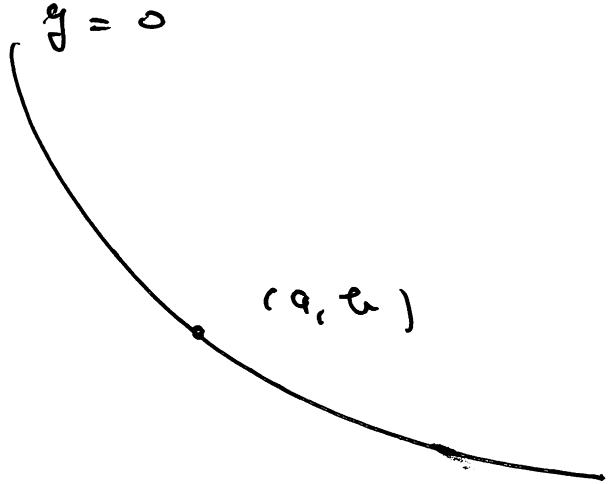
(5) $\int_0^{\frac{\pi}{2}} x \sin x \, dx = \int_0^{\frac{\pi}{2}} (-\cos x)' \, x \, dx$

$= -\left[x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 1 \cdot \cos x \, dx$

$= \left[\sin x\right]_0^{\frac{\pi}{2}} = 1 - 0 = 1$

$g_y(a, b) \neq 0$

$\longrightarrow$ 陰関数定理 $y = \varphi(x)$

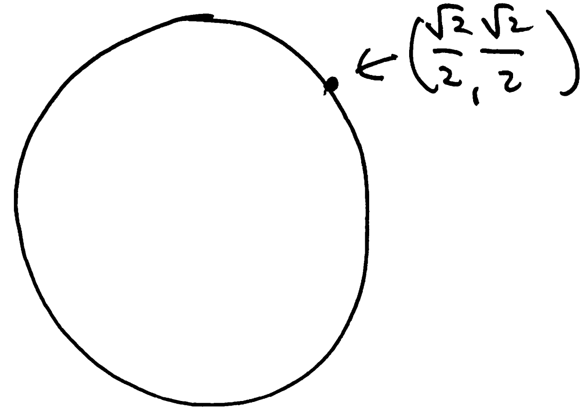


$$\varphi''(x) = \frac{1}{g_y^3} \begin{vmatrix} 0 & g_x & g_y \\ g_x & g_{xx} & g_{xy} \\ g_y & g_{yx} & g_{yy} \end{vmatrix}$$

フチ付き
Hesse 行列式.

<u>例</u> $g(x, y) = x^2 + y^2 - 1 = 0$

$$\varphi = \sqrt{1 - x^2}$$



$g_x = 2x, \quad g_y = 2y.$

$g_{xx} = 2, \quad g_{xy} = 0, \quad g_{yy} = 2$

$$\varphi'' = \frac{1}{(2y)^3} \begin{vmatrix} 0 & 2x & 2y \\ 2x & 2 & 0 \\ 2y & 0 & 2 \end{vmatrix}$$

$$\varphi''\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2\sqrt{2}} \begin{vmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2 & 0 \\ \sqrt{2} & 0 & 2 \end{vmatrix} = \frac{(\sqrt{2})^2}{2\sqrt{2}} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$y = \frac{\sqrt{2}}{2}$$

$$= \frac{1}{\sqrt{2}}(-2-2) = -2\sqrt{2}.$$

$$\varphi'' = -\frac{1}{\varphi^3} = -\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^3} = -2\sqrt{2}.$$

置換積分

$F'(x) = f(x)$

$y = F(x), \quad x = \varphi(t)$

$$\boxed{\frac{d}{dt} F(\varphi(t)) = f(\varphi(t))\,\varphi'(t)}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$
$$= f(x) \cdot \varphi'(t)$$
$$= f(\varphi(t)) \cdot \varphi'(t)$$

$$\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \Big[F(\varphi(t))\Big]_a^b$$

$$= F(B) - F(A)$$

$$= \int_A^B f(x)\,dx$$

$\varphi(b) = B$

$\varphi(a) = A$

まとめ

$$\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \int_A^B f(x)\,dx.$$

$$\int_0^1 \frac{t}{1+t^2}\,dt = \frac{1}{2}\int_0^1 \frac{(1+t^2)'}{(1+t^2)}\,dt$$

$(1+t^2)' = 2t$

$\frac{1}{2}(1+t^2)' = t$

$$= \frac{1}{2}\int_1^2 \frac{1}{x}\,dx$$

$f(x) = \frac{1}{x}$

$x = g(t) = 1+t^2$

$$= \frac{1}{2}\left[\log x\right]_1^2$$

$\boxed{(\log x)' = \frac{1}{x}}$

t	$0 \to 1$
x	$1 \to 2$

$$= \frac{1}{2}\log 2$$

$\log 1 = 0$

$$\int_0^1 \frac{t}{(1+t^2)^3}\,dt \quad \Leftarrow \quad f(x) = \frac{1}{x^3}, \quad x = g(t) = 1+t^2$$

$$\int_{-3}^{-1} \frac{dt}{(2t+1)^3} = \frac{1}{2}\int_{-3}^{-1} \frac{(2t+1)'}{(2t+1)^3}\,dt$$

$x = \varphi(t) = 2t+1$

$\varphi'(t) = 2$.

$$= \frac{1}{2}\int_{-5}^{-1} \frac{1}{x^3}\,dx$$

t	$-3 \nearrow -1$
x	$-5 \nearrow -1$

$$= \frac{1}{2}\left[-\frac{1}{2}\cdot\frac{1}{x^2}\right]_{-5}^{-1}$$

$$\left(\frac{1}{x^2}\right)' = -\frac{2}{x^3}$$

$$= -\frac{1}{4}\left(1 - \frac{1}{25}\right) = -\frac{6}{25}$$

$$\left(-\frac{1}{2}\cdot\frac{1}{x^2}\right)' = \frac{1}{x^3}$$

$$\int_{-3}^{-1} \frac{dx}{(2x+1)^3} = \int_{-5}^{-1} \frac{1}{t^3}\cdot\frac{1}{2}\,dt$$

$2x+1 = t$ となるようにする.

$$x = \varphi(t) = \frac{t-1}{2}$$

t	$-5 \nearrow -1$
x	$-3 \nearrow -1$

$= \cdots$

$$\int_A^B f(x)\,dx = \int_a^b f(\varphi(t))\,\varphi'(t)\,dt$$

例題5.10

$$\int_{-2}^{1} \frac{x}{\sqrt{x+3}}\,dx$$

$x+3=t$ となるように

$x = t-3.$

t	$1 \nearrow 4$
x	$-2 \nearrow 1$

$\varphi(t) = t-3$ とする

$\varphi'(t) = 1$

$$= \int_1^4 \frac{t-3}{\sqrt{t}}\,1\,dt$$

$$= \int_1^4 \left(\sqrt{t} - \frac{3}{\sqrt{t}}\right) dt$$

$$= \left[\frac{2}{3} t\sqrt{t} - 6\sqrt{t}\right]_1^4$$

$$= \frac{2}{3}(8-1) - 6(2-1)$$

$$= \frac{14}{3} - 6 = -\frac{4}{3}.$$

$(t\sqrt{t})' = \frac{3}{2}\sqrt{t}$

$\left(\frac{2}{3} t\sqrt{t}\right)' = \sqrt{t}$

$(\sqrt{t})' = \frac{1}{2}\frac{1}{\sqrt{t}}$

$(2\sqrt{t})' = \frac{1}{\sqrt{t}}$

$$\int_{-2}^{1} \frac{x}{\sqrt{x+3}}\,dx$$

$t = \sqrt{x+3}$.

$t \geq 0$

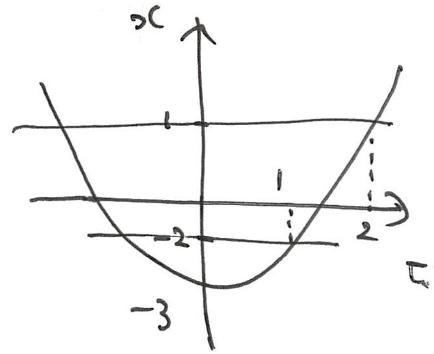
$t^2 = x+3$.

$x = t^2 - 3$

$\varphi(t) = t^2 - 3$

$\varphi'(t) = 2t$

$dx = 2t\,dt$



$$= \int_{1}^{2} \frac{t^2-3}{t} \cdot 2t\,dt$$

t	$1 \nearrow 2$
x	$-2 \nearrow 1$

$$= 2\int_{1}^{2} (t^2-3)\,dt$$

$$= 2\left[\frac{t^3}{3} - 3t\right]_{1}^{2} = \cdots = -\frac{4}{3}$$

$$\boxed{x = \varphi(t) \longrightarrow dx = \varphi'(t)\,dt}$$

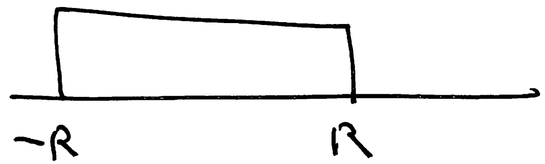
確率密度　$f(x) \geqq 0$　　$f : \mathbb{R} \longrightarrow \mathbb{R}$

$$\int_{-\infty}^{+\infty} f(x)\, dx = 1$$

全確率 $=1$

$$\parallel$$

$$\lim_{R \to +\infty} \int_{-R}^{R} f(x)\, dx$$



f : 確率密度

$$\int_{\alpha}^{\beta} f(x)\, dx = P(\alpha \leqq X \leqq \beta)$$

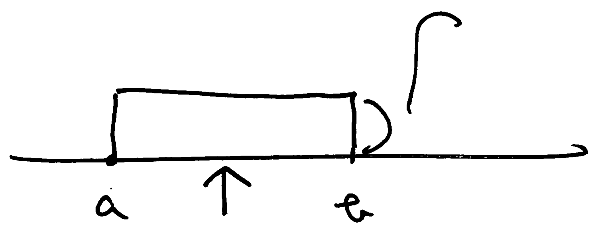
確率変数 X が $\alpha \leqq X \leqq \beta$ を満たす確率.

例)　$a < b$

一様分布.

$$f(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & x \notin [a, b] \end{cases}$$

$\frac{1}{b-a}$



$$\boxed{E[X] = \int_{-\infty}^{+\infty} x f(x)\, dx \quad X \text{の期待値.}}$$

$$E[X] = \int_a^b \frac{1}{b-a} x\, dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{2}(b^2 - a^2)/(b-a)$$

$$= \frac{1}{2}(a+b)$$

$$V[X] = \int_{-\infty}^{+\infty} (x-m)^2 f(x)\,dx$$

Xの分散

$$m = E[X]$$

$$= E[X^2] - E[X]$$

公式

$$= \int_{-\infty}^{+\infty} x^2 f(x)\,dx$$

一様分布について　$V[X] = ?$

I (1) $\displaystyle\int_0^1 (1+t^2)^4\, t\, dt$

(2) $\displaystyle\int_0^1 (1-t)^6\, dt$

(3) $\displaystyle\int_0^1 t(1-t)^5\, dt =$ 部分積分 $\{(1-t)^6\}' = \cdots$

(4) $\displaystyle\int_0^1 (2x+1)^4\, dx$

(5) $\displaystyle\int_0^1 \frac{t}{(1+t^2)^2}\, dt$

II

$g(x, y) = x^2 + xy + y^2 - 1 = 0$ (0, 1) において

$y = \varphi(x)$ と解く

$\varphi''(0)$ を [illegible]を行列式で計算.