calc 2015/12/02

$$f(x,y) = (x^2+y^2)^2 - 2(x^2-y^2)$$

$$f_x = 2(x^2+y^2)\cdot 2x - 4x.$$

$$= 4x(x^2+y^2-1) = 0$$

$$f_y = 2(x^2+y^2)\cdot 2y + 4y$$

$$= 4y\underbrace{\boxed{(x^2+y^2+1)}}_{\vee \atop 0} = 0.$$

$$f_x = 0 \iff x=0 \quad \text{or} \quad x^2+y^2=1$$

$$f_y = 0 \iff \boxed{y=0}$$

$$f_x = f_y = 0$$

$$\iff (y=0 \text{ and } x=0)$$

$$\text{or } (y=0 \text{ and } x^2+y^2=1)$$

$$\iff (x,y) = (0,0),\ (\pm 1, 0)$$

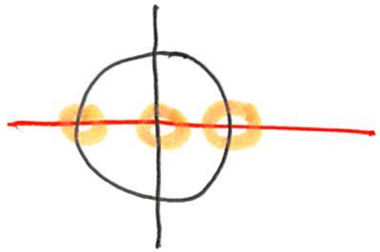
$$\frac{d}{dt}\left\{(t^2+a^2)^2\right\}$$

$$= \frac{du^2}{du}\cdot\frac{du}{dt} = 2u\cdot 2t$$

$$u = t^2+a^2$$

$$A\cdot B = 0$$

$$\iff A=0 \text{ or } B=0$$



$$P \text{ and } (Q \text{ or } R)$$

$$\equiv (P \text{ and } Q) \text{ or } (P \text{ and } R)$$

論理的に常に等しい

$f_x = 4x(x^2+y^2-1)$

$f_y = 4y(x^2+y^2+1)$

$(fg)' = f'g + fg'$

$f_{xx} = 4\{1(x^2+y^2-1) + x \cdot 2x\}$

$= 4(3x^2+y^2-1)$

f_{yx}

$\shortparallel$

$f_{xy} = 4x \cdot 2y = 8xy$

$f_{yy} = 4\{1(x^2+y^2+1) + y \cdot 2y\} = 4(x^2+3y^2+1)$

〃 $H(f)$ 2×2 .

$(0,0)$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -4 & 0 \\ 0 & 4 \end{vmatrix} = -16 < 0$$

$(0,0)$ では f は 極大でも極小でもない

$(\pm 1, 0)$

$$|H(f)| = \begin{vmatrix} 8 & 0 \\ 0 & 8 \end{vmatrix} = 64 > 0$$

$f_{xx}(\pm 1, 0) = 8 > 0$

より $(\pm 1, 0)$ で f は 極小.

$$f_x(P_0) = f_y(P_0) = 0$$

(I) (i) $\quad |H(f)(P_0)| = \begin{vmatrix} f_{xx}(P_0) & f_{xy}(P_0) \\ f_{yx}(P_0) & f_{yy}(P_0) \end{vmatrix} > 0$

(ii)

$$= f_{xx}(P_0) f_{yy}(P_0) - f_{xy}(P_0)^2$$

$f_{xx}(P_0) > 0$

$<$

大

$\Longrightarrow$ f は P_0 で極小.

(II) $\quad |H(f)(P_0)| < 0$

$\Longrightarrow$ f は P_0 で極大でも極小でもない.

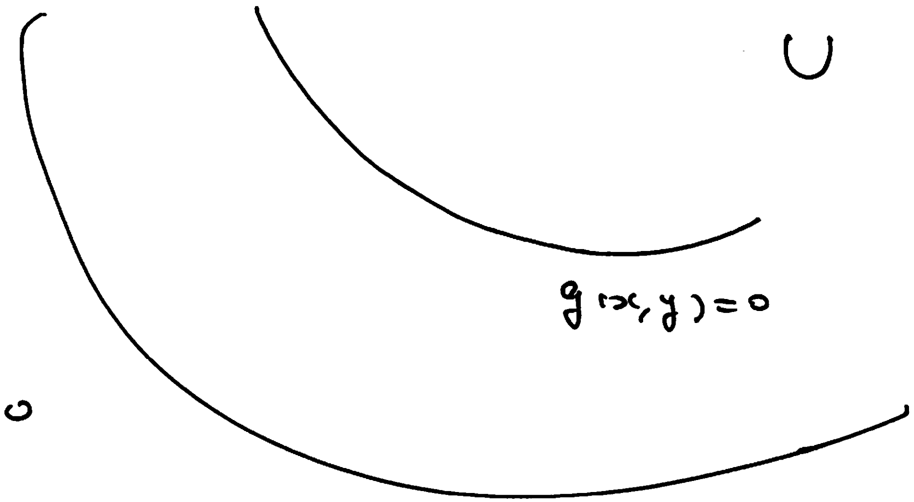
<u>283p ~</u>

$f : U \longrightarrow \mathbb{R}, \; g : U \longrightarrow \mathbb{R}$

$U \subset \mathbb{R}^2$ 開

$g(x, y) = 0$ の下で

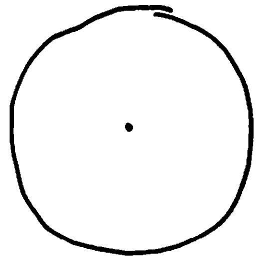
$z = f(x, y)$ を考える



<u>例118.10</u> $g(x, y) = x^2 + y^2 - 1 = 0$

の下で

$z = f(x, y) = xy.$



2財の効用最大化.

	1財	2財
購入量	x	y
価格.	p	q

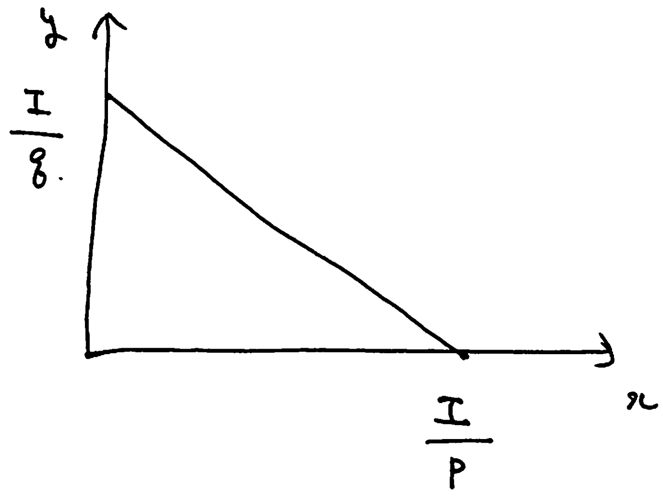
$\longrightarrow$ utility function 効用関数 $u(x,y)$

$p, q, I > 0$

$x, y > 0$

予算制約 $g(x,y) = I - px - qy = 0$ の下で

$z = u(x,y)$ を最大化.



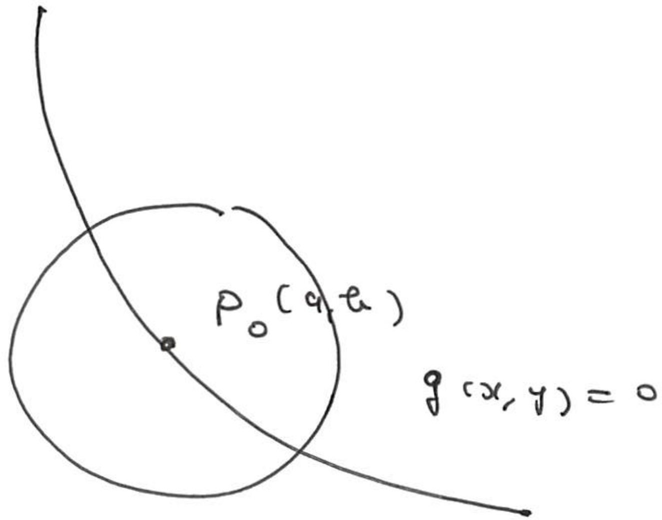
定理 8.23. (284 p.) 陰関数定理.

$g_y(P_0) \neq 0$ とする.

P_0 の近くで

$$g(x, y) = 0 \iff y = \varphi(x)$$

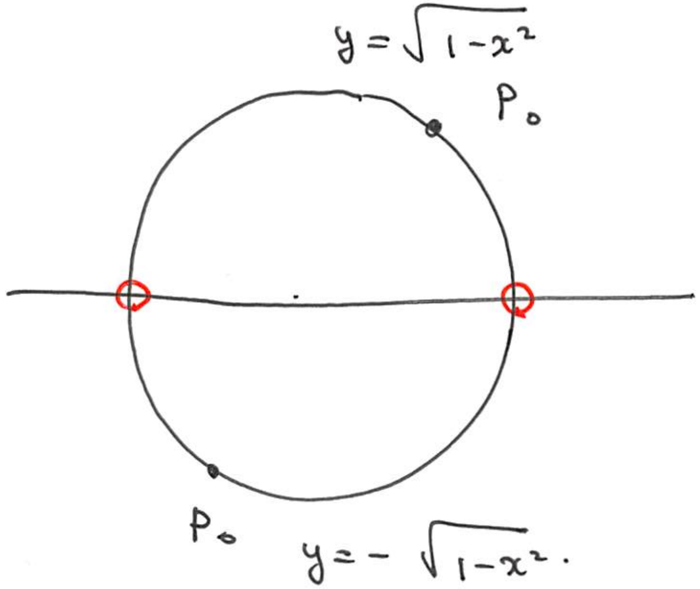
と書ける.



(例)

$$g(x, y) = x^2 + y^2 - 1 = 0$$

$$g_y = 2y \neq 0$$

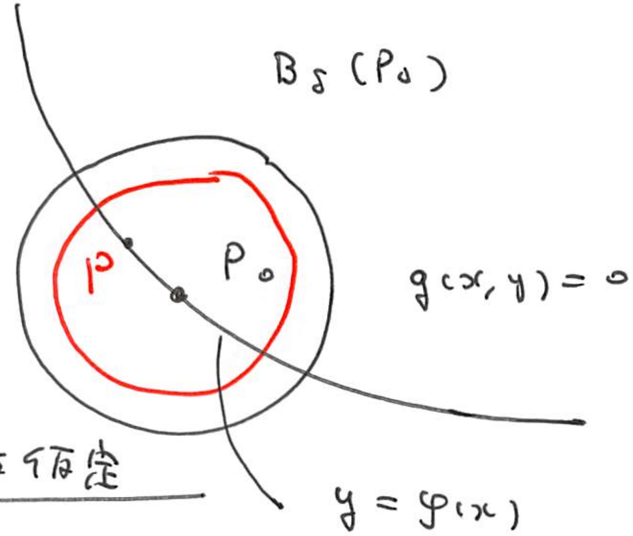


P_0 において 極大(小)

$P \in B_\delta(P_0)$

$g(P) = 0$

$\Longrightarrow \quad f(P_0) \overset{\leqq}{\geqq} f(P)$



P_0 で $\underset{0}{\underset{\parallel}{g(x,y)}}$ を満たし　<u>$g_y(P_0) \neq 0$ を仮定</u>

$F(t) = f(t, \varphi(t))$ とおく.　$\longrightarrow$ F は $t=a$ で 極大(小)

$$\longrightarrow \quad F'(a) = 0$$

$$\parallel$$

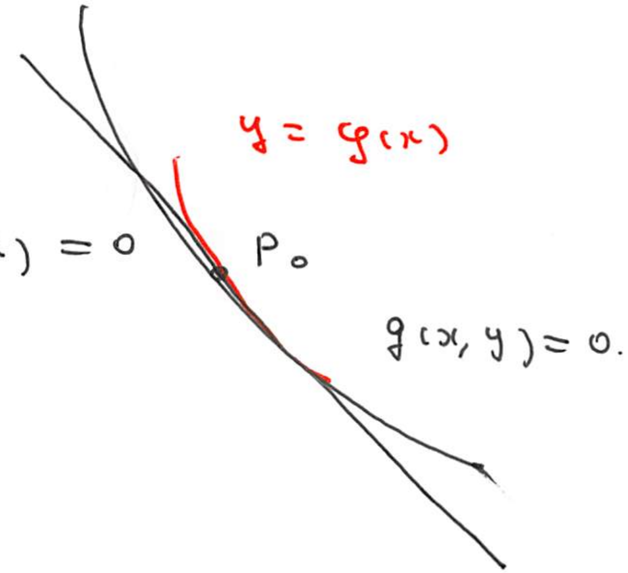
$$f_x(a,b) + f_y(a,b)\,\varphi'(a)$$

$$\boxed{\begin{array}{l} \text{chain rule} \\ F(t) = f(x(t), y(t)) \\ \longrightarrow F'(t) = f_x(x(t), y(t)) \cdot x'(t) \\ \qquad\qquad + f_y(\qquad) \cdot y'(t) \end{array}}$$

$$F'(t) = f_x(t, \varphi(t)) \cdot 1 + f_y(t, \varphi(t)) \cdot \varphi'(t)$$

$$\boxed{f_x(a,b) + f_y(a,b)\,\varphi'(a) = 0}$$

$\varphi'(a) = ?$　　接線の方程式.

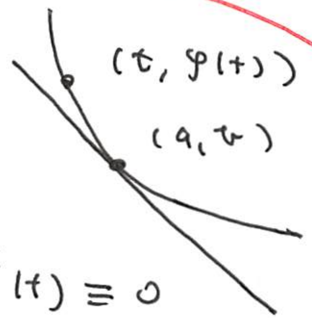


(その1)　　$g_x(a,b)(x-a) + g_y(a,b)(y-b) = 0$

$$\rightarrow\ y = \boxed{-\frac{g_x(P_0)}{g_y(P_0)}}(x-a) + b.$$

$= \varphi'(a)$

$$\boxed{\varphi'(a) = -\frac{g_x(P_0)}{g_y(P_0)}}$$



(その2)　　$g(t,\varphi(t)) \equiv 0.$

tで微分.　$g_x(t,\varphi(t))\cdot 1 + g_y(t,\varphi(t))\cdot\varphi'(t) \equiv 0$

t=aを代入　$g_x(a,b) + g_y(a,b)\,\varphi'(a) = 0$

$$f_x(a,b) + f_y(a,b)\left(-\frac{g_x(a,b)}{g_y(a,b)}\right) = 0$$

$= \lambda.$

$$\lambda = -\frac{f_y(a,b)}{g_y(a,b)}$$

$$\longrightarrow \quad \begin{cases} f_x(P_0) + \lambda g_x(P_0) = 0 \\ f_y(P_0) + \lambda g_y(P_0) = 0 \\ g(P_0) = 0. \end{cases} \qquad \leftarrow \quad F'(a) = 0$$

<u>定理</u> $g(x,y)=0$ の下で $z=f(x,y)$ を考える。

$P_0(a,b)$ で極大(小)

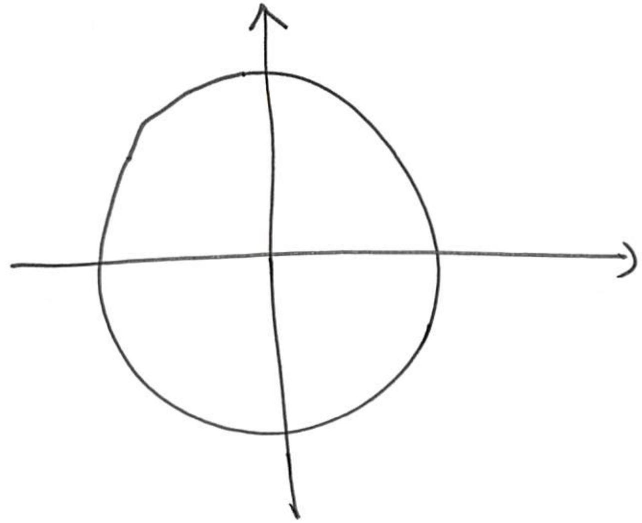
$$\Longrightarrow \quad \begin{cases} f_x(P_0) + \lambda g_x(P_0) = 0 \\ f_y(P_0) + \lambda g_y(P_0) = 0 \\ g(P_0) = 0 \end{cases}$$

を満たす $\lambda \in \mathbb{R}$ が存在.

(3)

$g(x, y) = x^2 + y^2 - 1 = 0$

$z = f(x, y) = xy.$



$g_x = 2x, \quad g_y = 2y.$

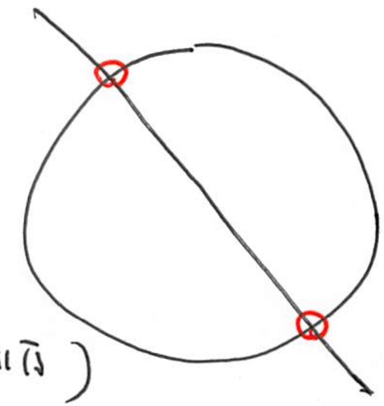
$f_x = y, \quad f_y = x.$

$$\begin{cases} y + \lambda \cdot 2x = 0 & \cdots ① \\ x + \lambda \cdot 2y = 0 & \cdots ② \\ x^2 + y^2 = 1 & \cdots ③ \end{cases}$$

①から $y = -2\lambda x$ これを②に代入 $x + 2\lambda(-2\lambda)x = 0$

$$x(1 - 4\lambda^2) = 0$$

(i) $x = 0$ とすると①より $y = 0$ となるが $(x, y) = (0, 0)$ は③を満たさない



(ii) $\lambda = \pm \frac{1}{2}.$

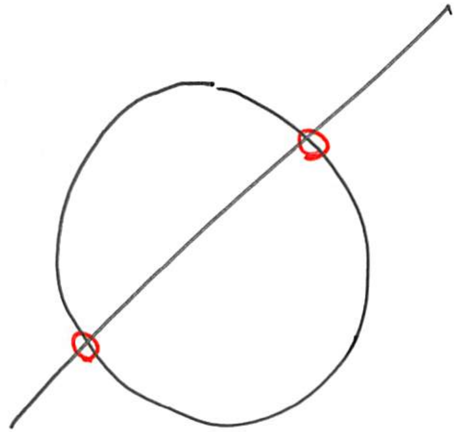
(I) $\lambda = \frac{1}{2}$ のとき ① ⟺ ② ⟺ $x + y = 0$

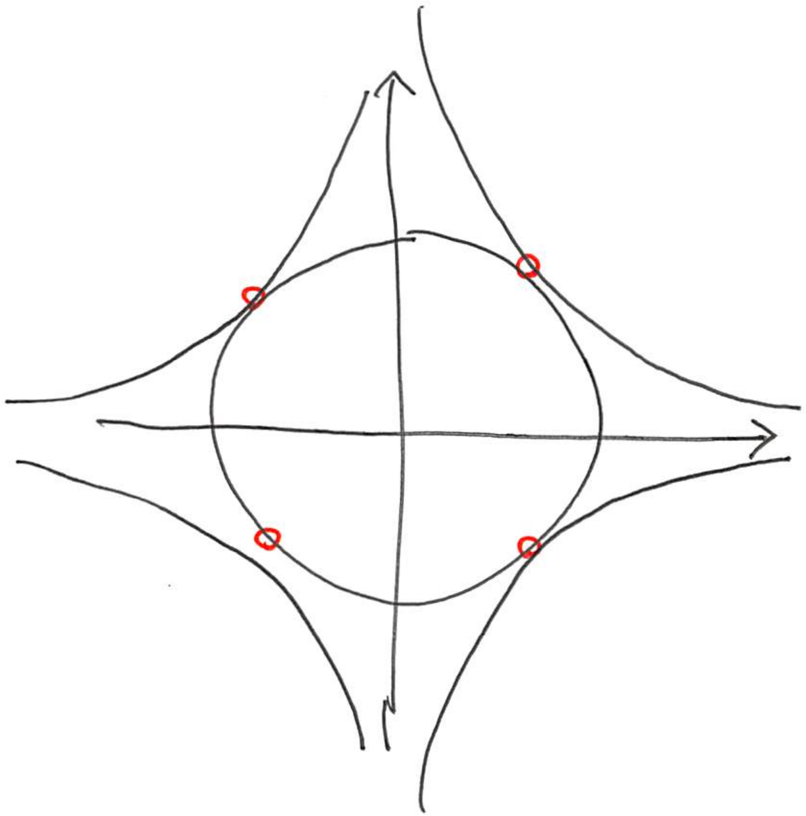
③を用いると $(x, y) = \left(\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}}\right)$ (複号同順)

Ⅱ $\lambda = -\frac{1}{2}$ のとき ① $\Leftrightarrow$ ② $\Leftrightarrow$ $x = y$.

③に代入して $(x, y) = \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

となる.





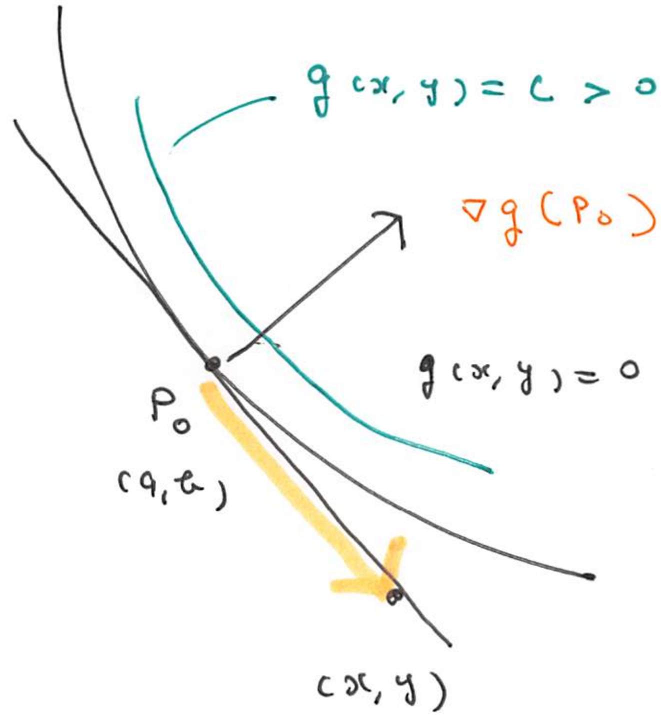
復習

接線の方程式

$$g_x(P_0)(x-a) + g_y(P_0)(y-b) = 0$$

$$\Updownarrow$$

$$\underbrace{\boxed{\begin{pmatrix} g_x(P_0) \\ g_y(P_0) \end{pmatrix}}}_{\nabla(g)(P_0)} \cdot \boxed{\begin{pmatrix} x-a \\ y-b \end{pmatrix}} = 0$$



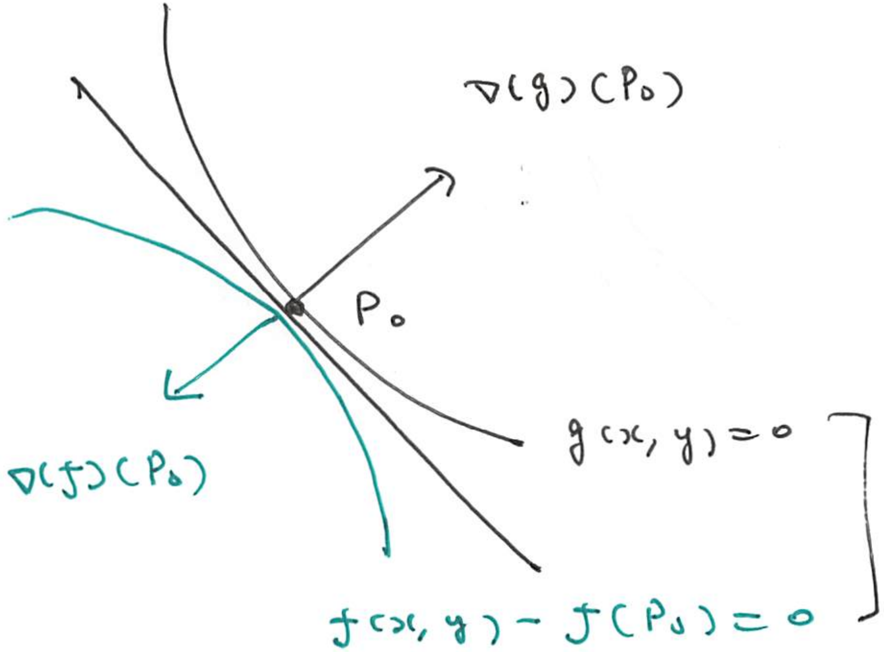
接線条件

$$\begin{cases} f_x(P_0) + \lambda g_x(P_0) = 0 \\ f_y(P_0) + \lambda g_y(P_0) = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{pmatrix} f_x(P_0) \\ f_y(P_0) \end{pmatrix} = -\lambda \begin{pmatrix} g_x(P_0) \\ g_y(P_0) \end{pmatrix}$$

$f(x,y) - f(P_0) = 0$ の P_0 における法ベクトル



2曲線はP_0で接線を共有

$\Longleftrightarrow$ 2曲線は P_0 で接する.

$g(x, y) = x^2 + 2y^2 - 1 = 0$ の下で

$z = f(x, y) = x + y$.

λ を求める