Calc 2015/11/25.

$A = \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix}$ を回転行列で対角化.

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1) 固有多項式を求める.

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda-2 & 2 \\ 2 & \lambda+1 \end{vmatrix} = (\lambda-2)(\lambda+1) - 4$$

$$\Phi_A(\lambda) = |\lambda I_2 - A|$$

$$= \lambda^2 - \lambda - 6$$

$$= (\lambda-3)(\lambda+2)$$

Aの固有値は $\lambda = 3, -2$.

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2) 固有ベクトルを求める.

$A\vec{r_1} = -2\vec{r_1}$

$\underline{\lambda = -2}$ $\quad \boxed{A\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}} \iff (-2I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$

$$\iff \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff 2x - y = 0.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (x \neq 0)$$

が固有ベクトル

$\lambda = 3$

$A\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix} \iff (3I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$ $A\vec{r_2} = 3\vec{r_2}$

$\iff \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff x + 2y = 0$

より $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $(y \neq 0)$ が固有ベクトル

回転行列.

$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\vec{r_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{r_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

とすると $R = (\vec{r_1}\ \vec{r_2})$ は回転行列.

$\frac{1}{\sqrt{5}}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$ ← 回転行列でない.

↓

$\frac{1}{\sqrt{5}}\begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}$ 回転行列.

$AR = (A\vec{r_1}\ A\vec{r_2}) = (-2\vec{r_1}\ 3\vec{r_2})$

$= (\vec{r_1}\ \vec{r_2})\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} = R\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$

$\boxed{(\vec{r_1}\ \vec{r_2})\begin{pmatrix} x \\ y \end{pmatrix} = x\vec{r_1} + y\vec{r_2}}$

Rは正則なので R^{-1}を左からかけて

$R^{-1}AR = R^{-1}R\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$ と対角化できる.

補足

$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$\parallel$$

$$2x^2-4xy-y^2$$

$$A=\begin{pmatrix}2&-2\\-2&-1\end{pmatrix}$$

R^{-1} 回転

$$=\left(R^{-1}A\begin{pmatrix}x\\y\end{pmatrix}, R^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$=\left(R^{-1}AR\cdot R^{-1}\begin{pmatrix}x\\y\end{pmatrix}, R^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$=\left(\begin{pmatrix}-2&0\\0&3\end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix}, \begin{pmatrix}\xi\\\eta\end{pmatrix}\right)=-2\xi^2+3\eta^2.$$

R 回転

$$\boxed{(R\vec{v}, R\vec{w})=(\vec{v}, \vec{w})}$$

$$R^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\xi\\\eta\end{pmatrix}\longleftrightarrow\begin{pmatrix}x\\y\end{pmatrix}=R\begin{pmatrix}\xi\\\eta\end{pmatrix}$$

$$=\xi\vec{r_1}+\eta\vec{r_2}$$

$$\begin{pmatrix}x'\\y'\end{pmatrix}=A\begin{pmatrix}x\\y\end{pmatrix}$$

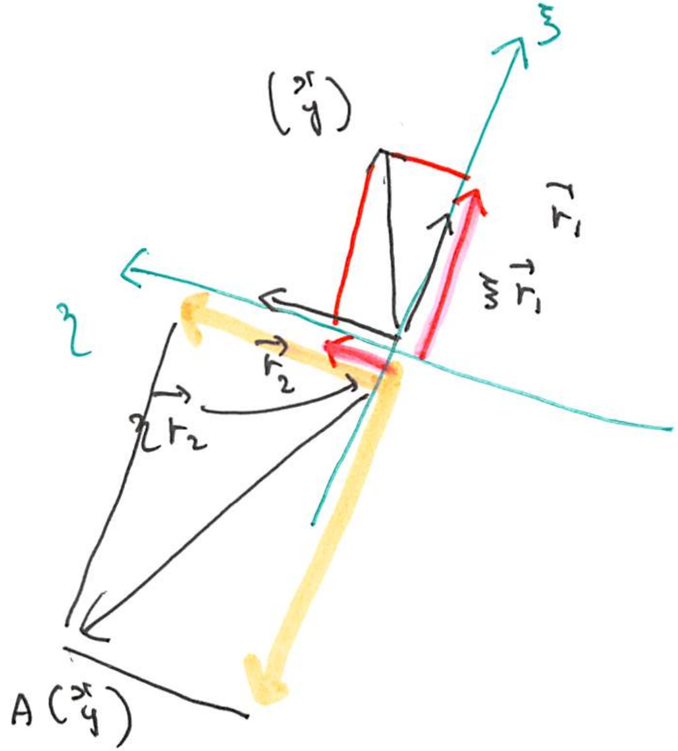
$$\updownarrow$$

$$\begin{pmatrix}\xi'\\\eta'\end{pmatrix}=R^{-1}\begin{pmatrix}x'\\y'\end{pmatrix}=R^{-1}A\begin{pmatrix}x\\y\end{pmatrix}$$

$$=R^{-1}AR\cdot R^{-1}\begin{pmatrix}x\\y\end{pmatrix}$$

$$=R^{-1}AR\begin{pmatrix}\xi\\\eta\end{pmatrix}$$

$$=\begin{pmatrix}-2&0\\0&3\end{pmatrix}\begin{pmatrix}\xi\\\eta\end{pmatrix}=\begin{pmatrix}-2\xi\\3\eta\end{pmatrix}$$



まとめ　対称行列の対角化．　$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$

① $\Phi_A(\lambda) = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = \lambda^2 - (a+b)\lambda + ab - c^2$

$D = (a+b)^2 - 4(ab - c^2) = (a-b)^2 + 4c^2 \geqq 0$

判別式

$\longrightarrow$ 固有値 $\in \mathbb{R}$

α, β

$\Updownarrow$

$(a-b)^2 = 4c^2 = 0$

$\Longleftrightarrow$ $a = b$ かつ $c = 0$

$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$$\boxed{\begin{array}{l} p, q \in \mathbb{R} \\ p, q \geqq 0 \\ p + q = 0 \Longrightarrow p = q = 0 \end{array}}$$

$\alpha = \beta \Longleftrightarrow A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a I_2$

対称行列について!!.

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② $\alpha \neq \beta$.　$A\vec{v}_1 = \alpha \vec{v}_1$, $A\vec{v}_2 = \beta \vec{v}_2 \Longrightarrow (\vec{v}_1, \vec{v}_2) = 0$.

③ A は回転行列で対角化できる　${}^{\exists}R$：回転

$R^{-1} A R = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ とできる

④ $\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \alpha \xi^2 + \beta \eta^2$　　$\begin{pmatrix} x \\ y \end{pmatrix} = R\begin{pmatrix} \xi \\ \eta \end{pmatrix}$

$\parallel$

$ax^2 + 2cxy + by^2$

2次形式の符号.

$R^{-1}AR = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} \xrightarrow[\text{行列式}]{} |R^{-1}AR| = \begin{vmatrix}\alpha & 0\\ 0 & \beta\end{vmatrix} = \alpha\beta.$

$= \underbrace{|R^{-1}|\cdot|A|\cdot|R|}_{=1} = |A|$

$A, B \in M_n(\mathbb{R}) \quad |AB| = |A|\cdot|B|$

$\downarrow$

$AB \in M_n(\mathbb{R})$

$|R^{-1}R| = |R^{-1}|\cdot|R|$

$\|$

$|I_2| = \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} = 1$

$\rightarrow \quad |R^{-1}| = \dfrac{1}{|R|}$

まとめ $\quad R^{-1}AR = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} \longrightarrow |A| = |\alpha\beta|$

$|A| \neq 0$ の場合を考える $\quad \alpha\beta \neq 0. \quad A = \begin{pmatrix}a & c\\ c & b\end{pmatrix}$

CT 242p.

(I) (a) $\alpha, \beta > 0 \iff |A| > 0,\ a > 0 \iff \left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) > 0 \quad \left(\begin{pmatrix}x\\y\end{pmatrix} \neq \vec{0}\right)$

(b) $\alpha, \beta < 0 \iff |A| > 0,\ a < 0 \iff \left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) < 0 \quad \left(\begin{pmatrix}x\\y\end{pmatrix} \neq \vec{0}\right)$

[illegible]
3正値内積.

(II) $\alpha\beta < 0 \iff |A| < 0$

$\iff \left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right)$ は正の値も負の値もとる.

$$A = \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix} \qquad R = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = (\vec{r}_1\ \vec{r}_2) \qquad R^{-1} A R = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$$

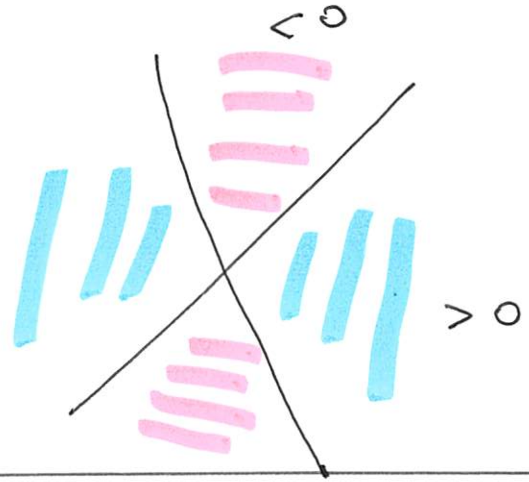
$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = -2\xi^2 + 3\eta^2 \qquad \begin{pmatrix} x \\ y \end{pmatrix} = R\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi\vec{r}_1 + \eta\vec{r}_2$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \vec{r}_1$$

$$= a t z\ (A\vec{r}_1, \vec{r}_1) = -2 < 0$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \vec{r}_2$$

$$(A\vec{r}_2, \vec{r}_2) = 3 > 0$$

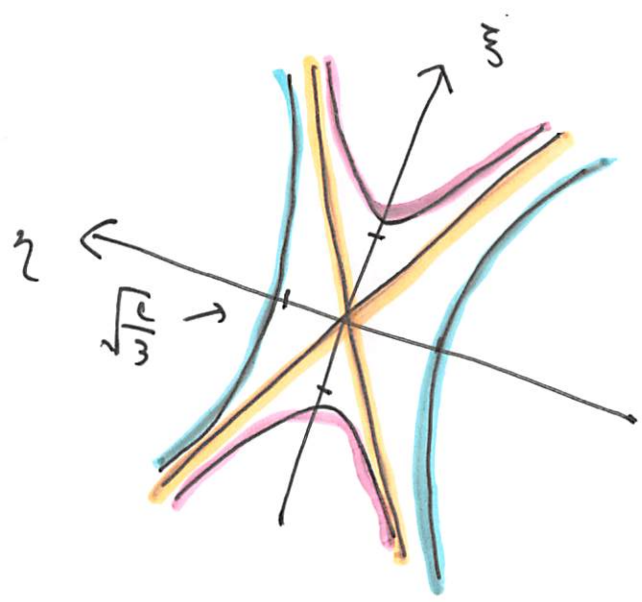


等高線 $\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = -2\xi^2 + 3\eta^2 = c$

$c = 0$

$\eta^2 = \frac{2}{3}\xi^2$

$\eta = \pm\sqrt{\frac{2}{3}}\,\xi$



$c > 0$

$\xi = 0$ のとき. $\eta = \pm\sqrt{\frac{c}{3}}$

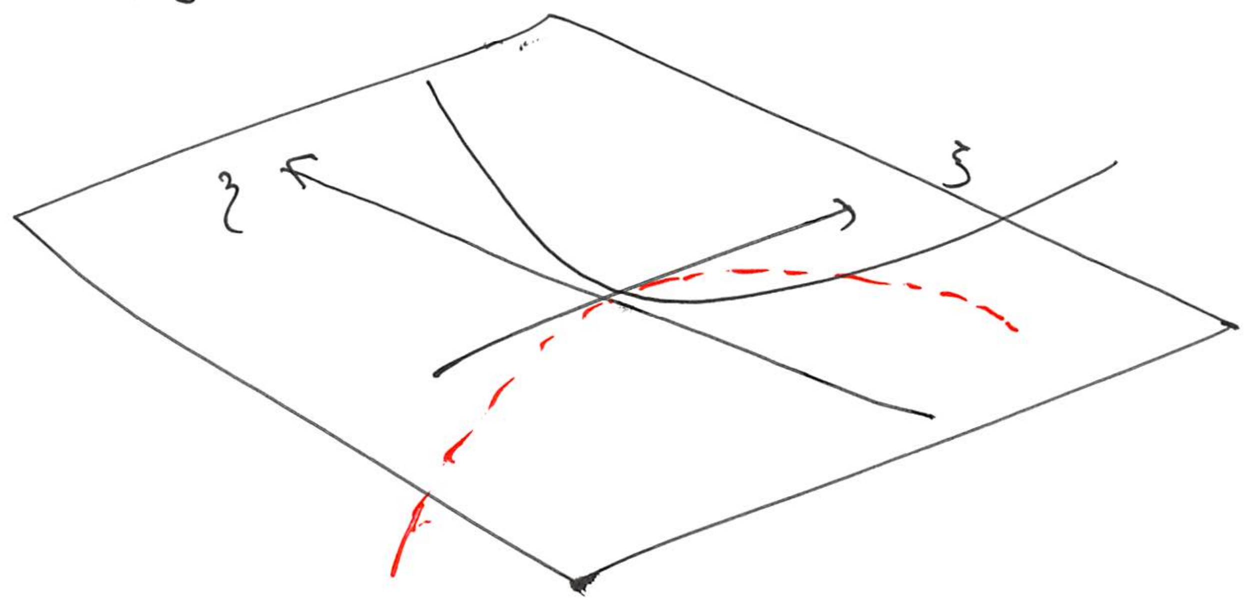
$c < 0$

$\eta = 0$ $\xi^2 = -\frac{c}{2}$

$\xi = \pm\sqrt{-\frac{c}{2}}$

$\xi = 0 \to 3\eta^2$

$\eta = 0 \to -2\xi^2$



(I) (a) $\alpha, \beta > 0$

(b) $\alpha, \beta < 0$ $\Big) \longrightarrow$ 楕円.

(II) $\alpha\beta < 0$ $\longrightarrow$ 双曲線

$\alpha, \beta > 0$.

$$\overbrace{\alpha \xi^2 + \beta \eta^2}^{\geq 0} = C.$$

1) $C < 0$ 空

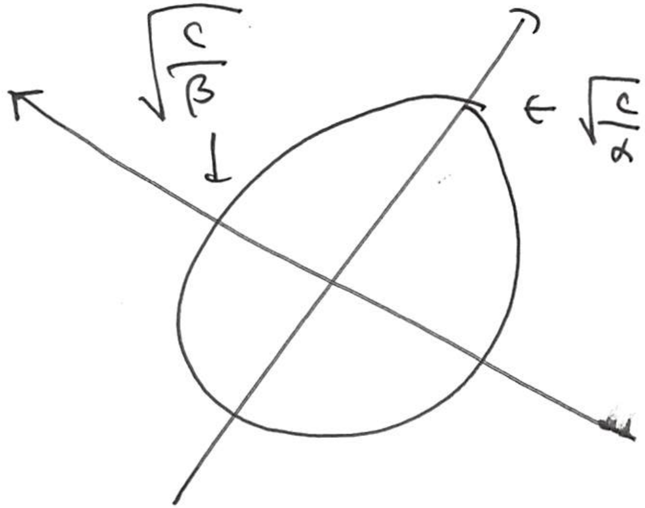
2) $C = 0$ 原点.

$$\underbrace{\alpha \xi^2}_{\geq 0} + \underbrace{\beta \eta^2}_{\geq 0} = 0 \longrightarrow \alpha \xi^2 = \beta \eta^2 = 0$$

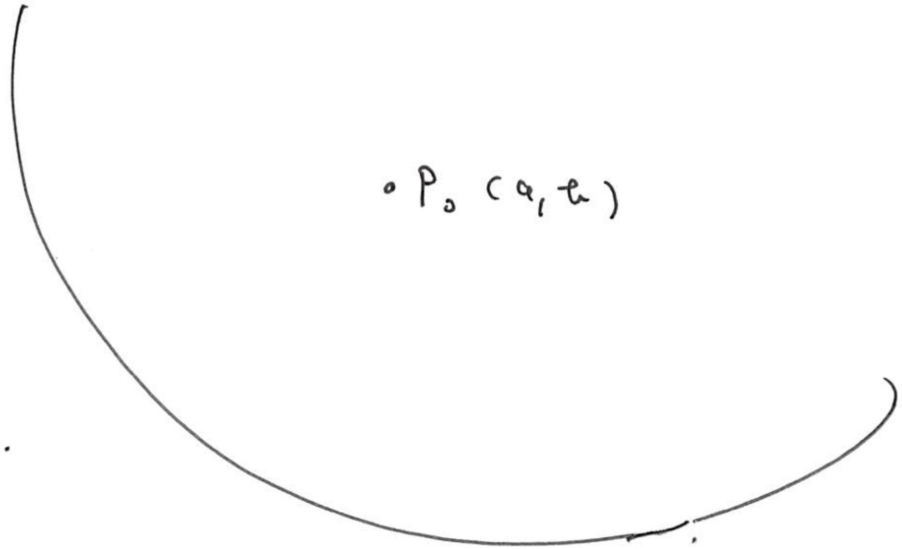
$$\longrightarrow \xi = \eta = 0 \longrightarrow x = y = 0.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

3) $C > 0$. $\alpha \xi^2 + \beta \eta^2 = C$ 楕円.



$f: U \longrightarrow \mathbb{R}$　　Youngの定理　　$U \subset \mathbb{R}^2$ 開



$f_x(P_0) = f_y(P_0) = 0$

"H 対称.

$$\left| \boxed{\begin{matrix} f_{xx}(P_0) & f_{xy}(P_0) \\ f_{yx}(P_0) & f_{yy}(P_0) \end{matrix}} \right| < 0$$

$\Longrightarrow$ P_0 で 極大でも 極小でも.

Hを回転行列 R で対角化.　　α, β 固有値

$$R^{-1} H R = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

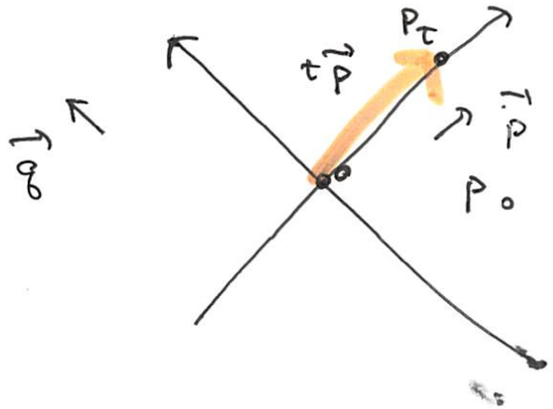
$|H| = \alpha\beta < 0$

$\alpha > 0, \beta < 0$ とする.

$\Updownarrow$

$R = (\vec{p} \ \ \vec{q})$ とすると

$$H\vec{p} = \alpha \vec{p}, \quad H\vec{q} = \beta \vec{q}$$



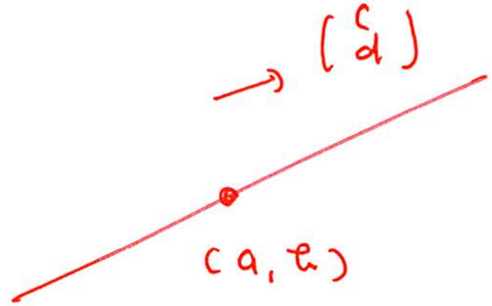
$$F(t) = f(\underbrace{P_0 + t\vec{p}}_{P_t})$$

$$G(t) = f(P_0 + t\vec{q})$$

方向微分 $F(t) = f(\underbrace{a + ct,\ b + dt}_{P_t})$

$$F'(t) = \left(\nabla f(P_t),\ \begin{pmatrix} c \\ d \end{pmatrix}\right)$$

$$F''(t) = \left(H(f)(P_t)\begin{pmatrix} c \\ d \end{pmatrix},\ \begin{pmatrix} c \\ d \end{pmatrix}\right)$$



$$\nabla(f) = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

$$H(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$$F'(t) = (\nabla(f)(P_t),\ \vec{p})$$

$$\longrightarrow F'(0) = (\nabla(f)(P_0),\ \vec{p}) = (\vec{0},\ \vec{p}) = 0$$

$$F''(t) = (H(f)(P_t)\vec{p},\ \vec{p})$$

$H\vec{p} = \alpha\vec{p}$

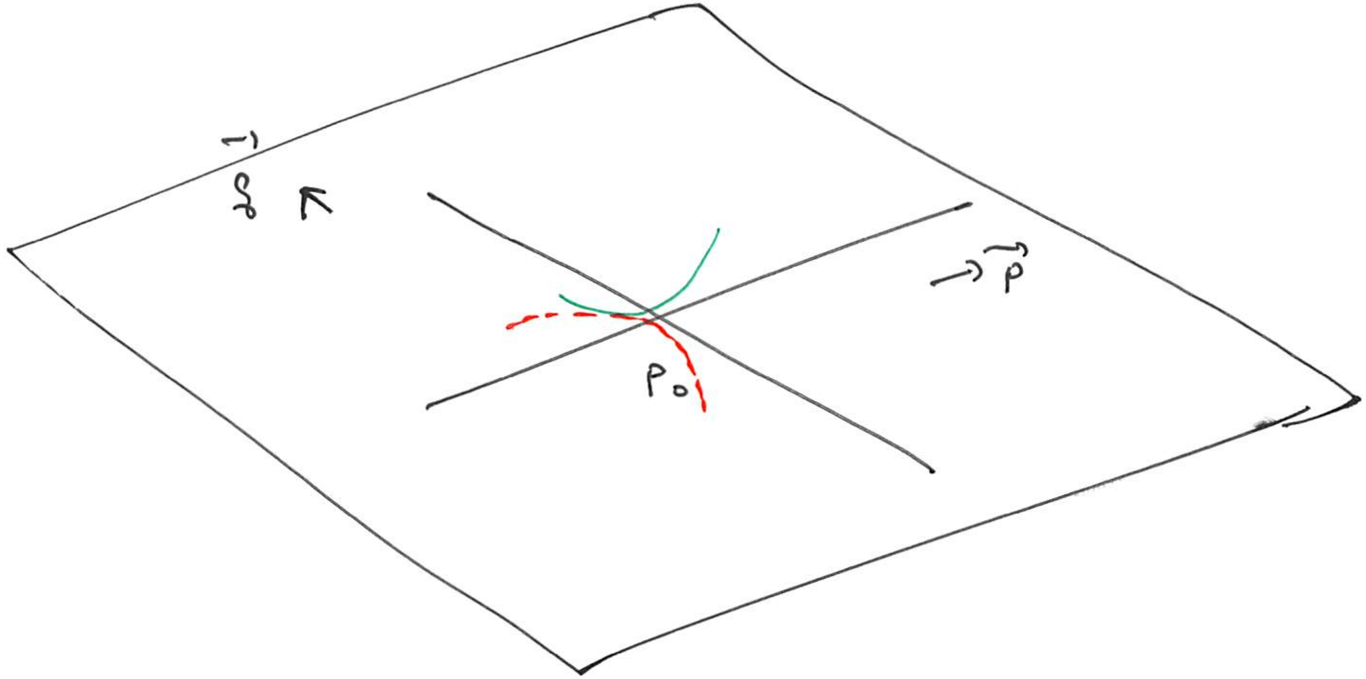
R回転

$$\longrightarrow F''(0) = (H\vec{p},\ \vec{p}) = (\alpha\vec{p},\ \vec{p}) = \alpha\overbrace{(\vec{p},\vec{p})}^{1}$$

$$= \alpha > 0$$

$$G'(0) = (\nabla(f)(P_0), \vec{q}) = (\vec{0}, \vec{q}) = 0$$

$$G''(0) = (H\vec{q}, \vec{q}) = (\beta\vec{q}, \vec{q}) = \beta \|\vec{q}\|^2 = \beta < 0$$



$$z = (x^2+y^2)^2 - 2(x^2-y^2)$$

の停留点を求めて極大・極小を判定.