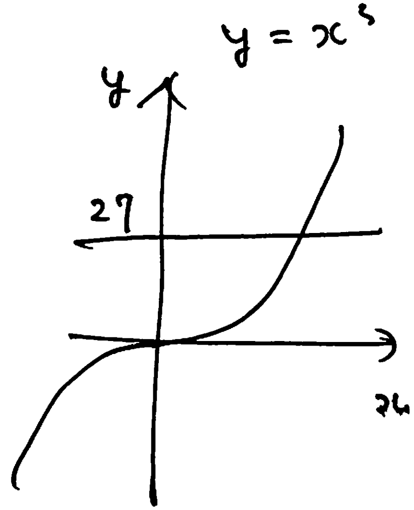
$z = x^3 + y^3 - 9xy + 27$ の停留点を求める.

$$\begin{cases} z_x = 3x^2 - 9y = 0 \\ z_y = 3y^2 - 9x = 0 \end{cases} \iff \begin{cases} x^2 = 3y \quad \cdots ① \\ y^2 = 3x \quad \cdots ② \end{cases}$$



① より $y = \dfrac{x^2}{3}$ これを ② に代入

$$\frac{x^4}{9} = 3x \rightsquigarrow x^4 = 27x.$$

$$\iff \quad x = 0 \quad \text{or} \quad x^3 = 27$$

$$\iff \quad x = 0 \quad \text{or} \quad x = 3$$

$x = 0$ のとき ① に代入して $y = 0$

$x = 3$ のとき ① に代入して $y = 3$

よって $(x, y) = (0, 0), (3, 3)$ が停留点

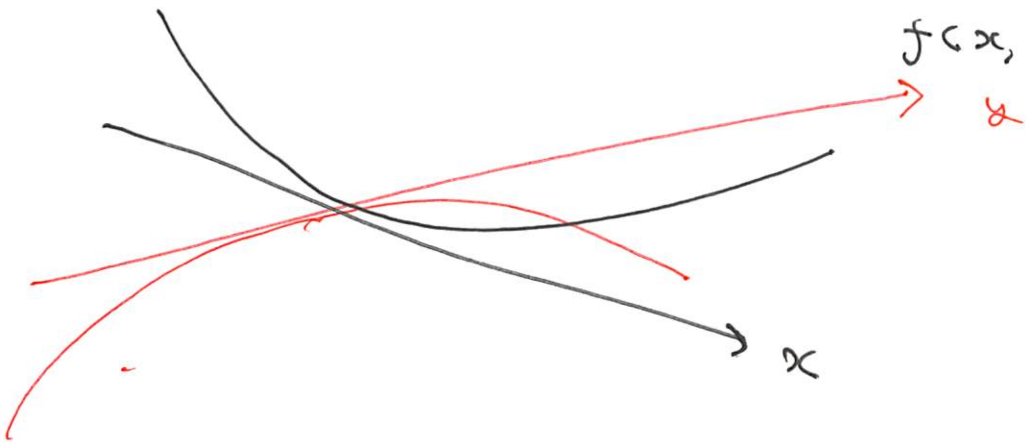
U ($\subset \mathbb{R}^2$) 開　$f : U \longrightarrow \mathbb{R}$　$(a,b) \in U$

$$f_x(a,b) = f_y(a,b) = 0 \quad \leftarrow \quad (a,b) \text{ は停留点}$$

$$\Uparrow$$

f が (a,b) で極大・極小

CT269.　$f(x,y) = x^2 - y^2$　$\begin{cases} f_x = 2x = 0 \\ f_y = -2y = 0 \end{cases} \Longleftrightarrow (x,y) = (0,0)$



$f(x,0) = x^2$, $f(0,y) = -y^2$

$(0,0)$ では極大でも極小でもない.

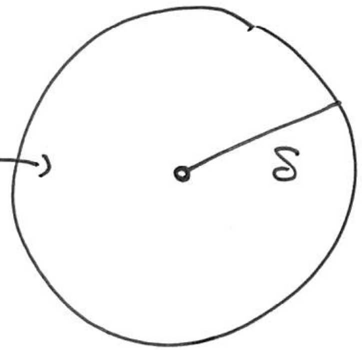
$(0,0)$ で極小ならば

大

ある $\delta > 0$ に対し

$$f(x,y) \geqq f(0,0)$$

$\leqq$



$((x,y) \in B_\delta(0,0))$

開円盤.

例1 $f(x, y) = x^2 - y^2$

$f_x = 2x,\ f_y = -2y.$

$f_{xx} = 2,\ f_{xy} = 0,\ f_{yx} = 0,\ f_{yy} = -2.$

$(0, 0)$ において

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0$$

$\longrightarrow$ $(0, 0)$ で極大でも極小でもない

例2 $z = x^3 + y^3 - 9xy + 27$

$z_x = 3x^2 - 9y,\ z_y = 3y^2 - 9x.$

$z_{xx} = 6x,\ z_{xy} = -9,\ z_{yx} = -9,\ z_{yy} = 6y.$

$(0, 0)$

$$\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -9 \\ -9 & 0 \end{vmatrix} = -81 < 0$$ $\longrightarrow$ 極大でも極小でもない

$(3, 3)$

$H = \begin{vmatrix} 18 & -9 \\ -9 & 18 \end{vmatrix} = 81 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 81 \cdot (4-1) = 243 > 0$

$9\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$9\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$z_{xx}(3, 3) = 18 > 0$ $\longrightarrow$ $(3, 3)$ で極小

定理

$f : U \longrightarrow \mathbb{R}$　　$U \ (\subset \mathbb{R}^2)$ 開.

$$\underset{(a,b)}{\overset{\downarrow}{U}}$$

だいたいの場合

Youngの定理.

$$f_x(a,b) = f_y(a,b) = 0$$

1)
$$\begin{vmatrix} f_{xx}(a,b) & \boxed{f_{xy}(a,b)} \\ \boxed{f_{yx}(a,b)} & f_{yy}(a,b) \end{vmatrix} > 0$$

$$f_{xx}(a,b) \ \underset{<}{>} \ 0$$

$\Longrightarrow$ (a,b) で極小 大

2)
$$\begin{vmatrix} & \end{vmatrix} < 0 \Longrightarrow \text{極大でも極小でもない}$$

$$(f_x)_x = f_{xx}$$

$$(f_x)_y = f_{xy}$$

$$(f_y)_x = f_{yx}$$

$$(f_y)_y = f_{yy}$$

2階の偏微分

$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ $\quad \left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = ax^2 + 2cxy + by^2.$ Aの2次形式.

Quadratic form.

定理.

244p.

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) > 0 \quad \left(\begin{pmatrix} x \\ y \end{pmatrix} \neq \vec{0}\right) \iff \text{Aの固有値 } \alpha, \beta > 0$$

$$\iff a > 0, \ \begin{vmatrix} a & c \\ c & b \end{vmatrix} = ab - c^2 > 0$$

CTにない。

$\Rightarrow$

$a > 0$ は何故か?

$$0 < \left(A\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = a \cdot 1^2 + 2c\,1 \cdot 0 + b \cdot 0^2 = a$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \vec{0}$

$$ax^2 + 2cxy + by^2 = a\left(x + \frac{c}{a}y\right)^2 + by^2 - a \cdot \frac{c^2}{a^2}y^2$$

$$= a\left(x + \frac{c}{a}y\right)^2 + \frac{ab - c^2}{a} \cdot y^2.$$

$$\underset{\vee \atop 0}{\left(A\begin{pmatrix} -\frac{c}{a} \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{c}{a} \\ 1 \end{pmatrix}\right)} = \frac{ab - c^2}{a} \ \underset{a > 0}{\leadsto} \ ab - c^2 > 0$$

($\Leftarrow$) $a>0,\ ab-c^2>0$ と仮定

$(x,y)\neq(0,0)$

$$ax^2+2cxy+by^2 = \underset{\vee\atop 0}{a}\,\underbrace{\left(x+\frac{c}{a}y\right)^2}_{\geqq\,0} + \underset{\vee\atop 0}{\frac{ab-c^2}{a}}\,\underset{\geqq\atop 0}{y^2} \underset{=}{\overset{\Updownarrow}{>}} 0$$

$\alpha,\beta\geqq 0$ とする

$\alpha+\beta=0 \Longrightarrow \alpha=\beta=0$

$\Longleftrightarrow a\left(x+\frac{c}{a}y\right)^2=\frac{ab-c^2}{a}y^2=0$

$\Longleftrightarrow x+\frac{c}{a}y=y=0$

$\Longleftrightarrow x=y=0$

定理 1) の仮定. $\begin{vmatrix} f_{xx}(a,b) & \boxed{f_{xy}(a,b)} \\ \boxed{f_{yx}(a,b)} & f_{yy}(a,b) \end{vmatrix}>0$ かつ $f_{xx}(a,b)>0$

$P_0(a,b)$

$\leadsto$ 定理

$$f_{xx}(P_0)\xi^2+2f_{xy}(P_0)\xi\eta+f_{yy}(P_0)\eta^2>0$$

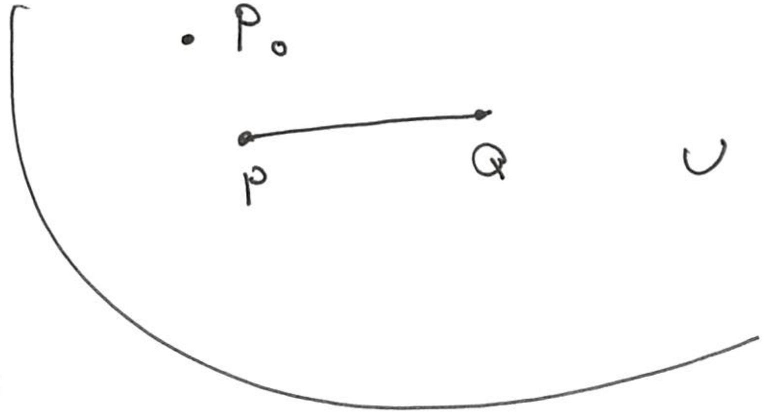
$$\left(\begin{pmatrix}\xi\\ \eta\end{pmatrix}\neq\vec{0}\right)$$

定理　$U \ (\subset \mathbb{R}^2)$ 開，凸

（最大・最小）

$P, Q \in U$

$\Longrightarrow \overline{PQ} \subset U$



$f : U \longrightarrow \mathbb{R}$

1)　$f_{xx}(P) \overset{<}{>} 0 \quad (P \in U)$

2)　$\begin{vmatrix} f_{xx}(P) & \boxed{f_{xy}(P)} \\ \boxed{f_{yx}(P)} & f_{yy}(P) \end{vmatrix} > 0 \quad (P \in U)$

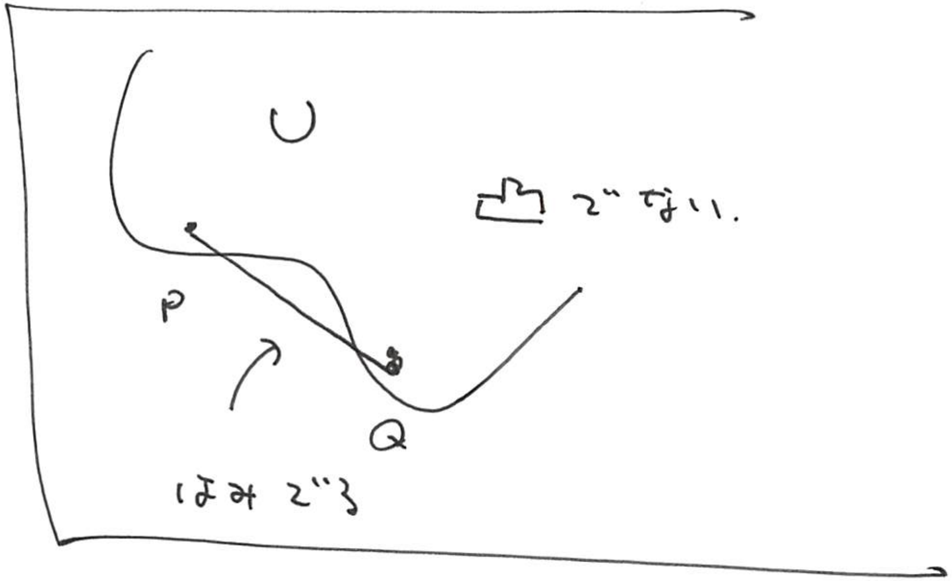
3)　$P_0 \in U$ において

$f_x(P_0) = f_y(P_0) = 0$

$\Longrightarrow f(P) \underset{<}{>} f(P_0)$

$(P \neq P_0)$

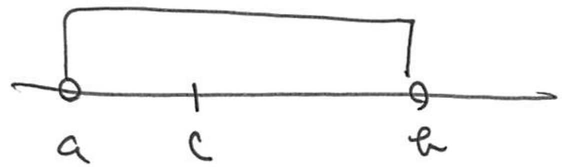
$(P \in U)$



(I) 1) から 2) $\rightsquigarrow$ $p \in U$, $\begin{pmatrix} \xi \\ \eta \end{pmatrix} \neq \vec{0}$

$$f_{xx}(p)\xi^2 + 2f_{xy}(p)\xi\eta + f_{yy}(p)\eta^2 > 0.$$

(II) 1変数の定理 CT 120p 定理4.8
145p — 4.15



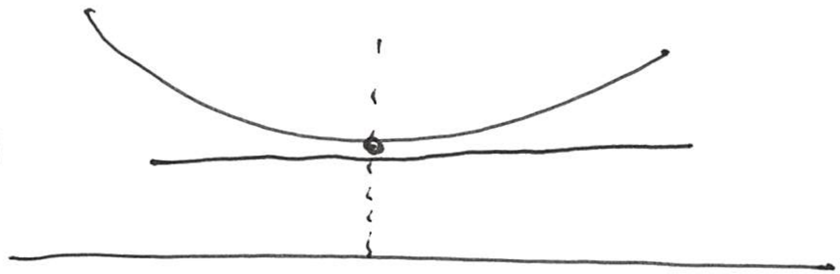
$$F : (a, b) \longrightarrow \mathbb{R}$$

$$F''(t) > 0 \quad (t \in (a, b))$$

$$\Longrightarrow \quad F(t) > F(c) + \underset{\substack{\| \\ 0}}{F'(c)}(t - c) \quad (t \neq c)$$

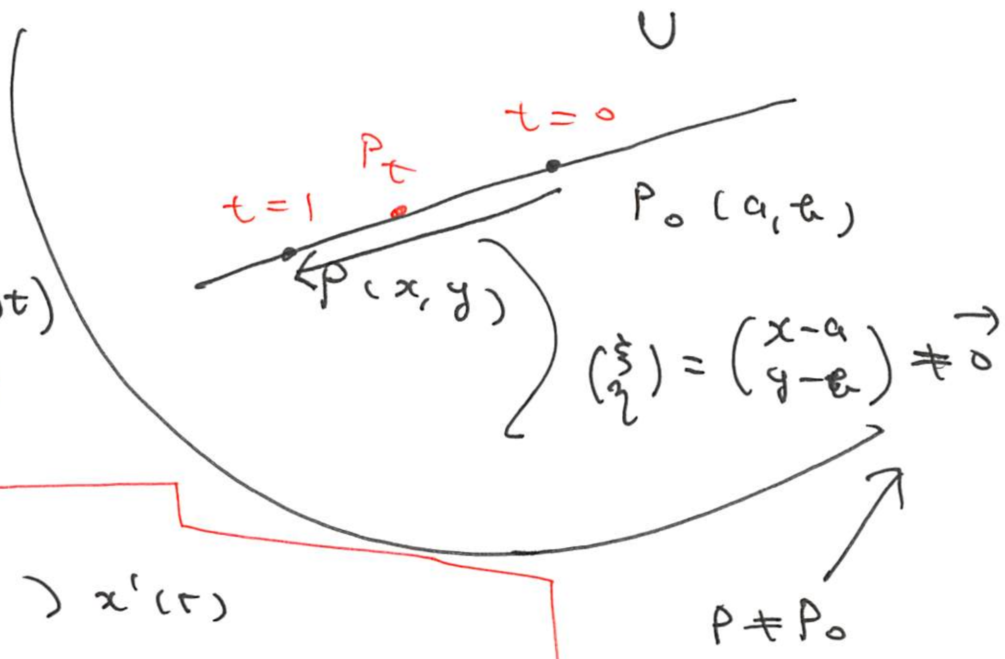
特に $F'(c) = 0$

$$F(t) > F(c) \quad (t \neq c)$$



最大・最小の証明.

$P \in U$ で $P \neq P_0$ とする.



$$F(t) = f(a+(x-a)t,\ b+(y-b)t)$$

$$= f(a+\xi t,\ b+\eta t) \leftarrow P_t$$

Chain Rule

$$\frac{d}{dt} f(x(t), y(t)) = f_x(\quad)\, x'(t) + f_y(\quad)\, y'(t)$$

$$F'(t) = f_x(a+\xi t,\ b+\eta t)\cdot \xi + f_y(\quad)\cdot \eta.$$

ξ, η は t について定数.

$$F''(t) = \xi\left(f_{xx}(P_t)\cdot \xi + f_{xy}(P_t)\cdot \eta\right) + \eta\left(f_{yx}(P_t)\,\xi + f_{yy}(P_t)\,\eta\right)$$

$$= f_{xx}(P_t)\,\xi^2 + 2 f_{xy}(P_t)\,\xi\eta + f_{yy}(P_t)\,\eta^2$$

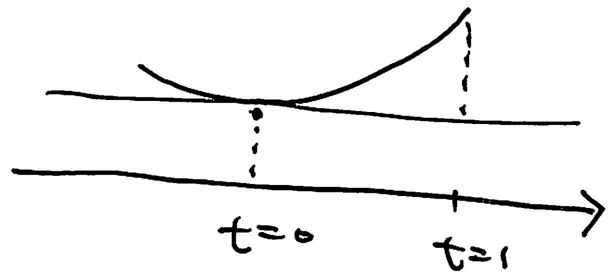
$\begin{pmatrix}\xi\\ \eta\end{pmatrix} \neq \vec{0}$, $P_t \in U$ ← Uは凸

→ $F''(t) > 0$

$F'(0) = f_x(a,b)\cdot\xi + f_y(a,b)\eta = 0\cdot\xi + 0\cdot\eta = 0$

→ 1変数の定理

$F(0) < F(1)$



→ $f(a,b) < f(x,y)$

証明