I (1) $z = x^2 - xy + y^2 - 2x + 4y$ の停留点.

$z_x = z_y = 0$ の点.

$$z_x = 2x - y \cdot 1 + 0 - 2 + 0$$
$$= 2x - y - 2 = 0$$
$$z_y = 0 - x \cdot 1 + 2y - 0 + 4$$
$$= -x + 2y + 4 = 0$$

$$z_x = z_y = 0 \iff \begin{cases} 2x - y = 2 \\ -x + 2y = -4. \end{cases}$$

$$D = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0$$

$$x = \frac{1}{3} \begin{vmatrix} 2 & -1 \\ -4 & 2 \end{vmatrix} = 0 \quad \longleftarrow \quad \begin{pmatrix} 2 \\ -4 \end{pmatrix} /\!/ \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$y = \frac{1}{3} \begin{vmatrix} 2 & 2 \\ -1 & -4 \end{vmatrix} = \frac{1}{3}(-8 + 2) = -2$$

停留点は $(x, y) = (0, -2)$

$$\begin{cases} ax + by = \alpha \\ cx + dy = \beta. \end{cases}$$

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$
$$= ad - bc$$

$$\begin{cases} x = \dfrac{1}{D} \begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix} \\[2ex] y = \dfrac{1}{D} \begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix} \end{cases}$$

(2) $z = \dfrac{1}{x^2+y^2}$

$$z_x = -\frac{1}{(x^2+y^2)^2} \cdot 2x$$

$$= -\frac{2x}{(x^2+y^2)^2}$$

$$z_y = -\frac{2y}{(x^2+y^2)^2}$$

$u = x^2+y^2$ とおくと

$z = \dfrac{1}{u}$

$\left(\dfrac{1}{u}\right)' = -\dfrac{1}{u^2}$

II (1)

$$\left(\begin{pmatrix} \boxed{a \quad c} \\ \boxed{c \quad b} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\begin{pmatrix} ax + cy \\ cx + by \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$= (ax + cy)x + (cx + by)y$$

$$= ax^2 + cxy + cxy + by^2$$

$$= ax^2 + \underline{\underline{2}}cxy + by^2 \quad \leftarrow A = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \text{が定める2次形式.}$$

$$z = x^2 - xy + y^2 - 2x + 4y \quad \leftarrow x, y \text{ の 2次式.}$$

$$= \left(\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) + \boxed{(-2 \quad 4)\begin{pmatrix} x \\ y \end{pmatrix}}$$

$$\boxed{(-2 \quad 4)\begin{pmatrix} x \\ y \end{pmatrix}} = \left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$(a_1\ a_2 \cdots a_n)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$\left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}\right)$$

$$\boxed{\vec{a}, \vec{b} \in \mathbb{R}^n \qquad {}^t\vec{a}\,\vec{b} = (\vec{a}, \vec{b})}$$

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \rightsquigarrow {}^t\vec{a} = (a_1\ a_2 \cdots a_n)$$

$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ は対称行列.

$${}^t\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

$${}^tA = A.$$

一般に

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow {}^tA = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

(2) $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = |A|\cdot I_2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ に対して $|A| = ad-bc$

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = |A|\, I_2$

tilde

$\tilde{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ Aの余因子行列.

$$A\tilde{A} = \tilde{A}A = |A|\, I_2$$

$\longrightarrow$ $|A| \neq 0$ のとき.

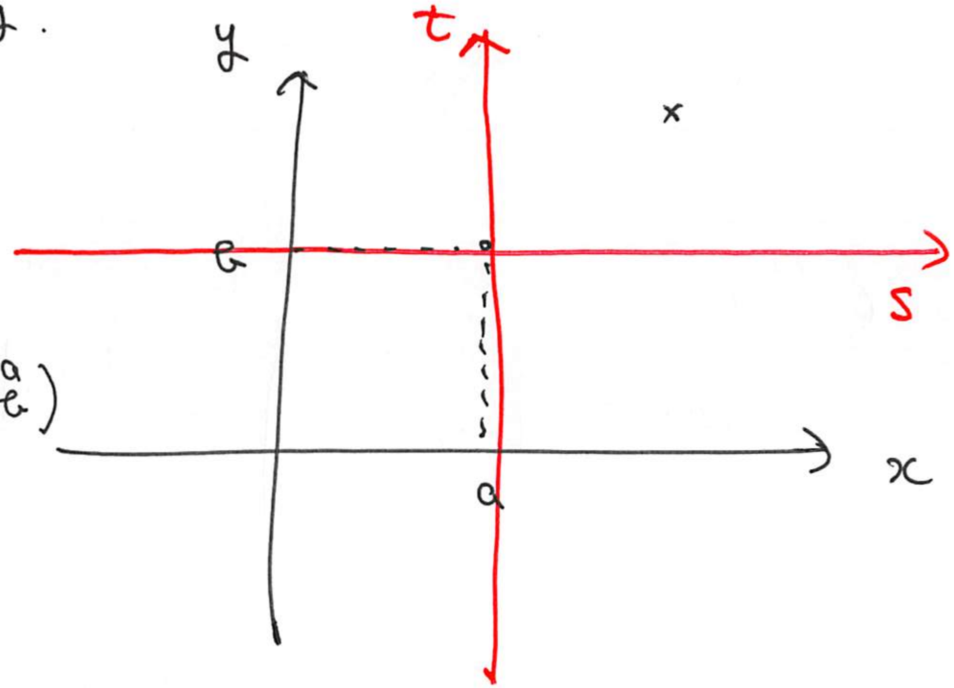
$$A\cdot\frac{1}{|A|}\tilde{A} = \frac{1}{|A|}\tilde{A}\cdot A = I_2$$

定理 A : 2×2 $|A| \neq 0 \Rightarrow$ A : 正則.

$z = x^2 - xy + y^2 - 2x + 4y.$

平行移動の座標変換

$$\begin{cases} s = x - a \\ t = y - b. \end{cases} \rightarrow \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}$$



$$= \left(\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) + \left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

a, b を選ぶ $\longrightarrow$ $(a, b) = (0, -2)$ 停留点.

$$= s^2 - st + t^2 - 4.$$

$\boxed{\text{定理} \quad A : 2\times 2 \quad \vec{v}, \vec{w} \in \mathbb{R}^2 \quad A\vec{v} \in \mathbb{R}^2 \quad (A\vec{v}, \vec{w}) = (\vec{v}, {}^tA\vec{w})}$

← 来週.

$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{w} = \begin{pmatrix} s \\ t \end{pmatrix}$, $\vec{v}_0 = \begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \vec{w} = \vec{v} - \vec{v}_0$

$A = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$ $\qquad \vec{v} = \vec{w} + \vec{v}_0$

$\vec{\beta} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

$$z = (A\vec{v}, \vec{v}) + (\vec{\beta}, \vec{v})$$

$$= (\boxed{A(\vec{w} + \vec{v}_0)}, \vec{w} + \vec{v}_0) + (\vec{\beta}, \vec{w} + \vec{v}_0)$$

$$\parallel$$

$$A\vec{w} + A\vec{v}_0$$

$$= (A\vec{w}, \vec{w}) + \boxed{(A\vec{w}, \vec{v}_0)} + (A\vec{v}_0, \vec{w}) + (A\vec{v}_0, \vec{v}_0)$$

$$+ (\vec{\beta}, \vec{w}) + (\vec{\beta}, \vec{v}_0)$$

$\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$

$\parallel$

A : 対称.

${}^tA = A$

$$= (\vec{w}, {}^tA\vec{v}_0)$$

$$= (\vec{w}, A\vec{v}_0)$$

$$= (A\vec{v}_0, \vec{w})$$

$$= (A\vec{w},\vec{w}) + 2(A\vec{v}_0,\underline{\underline{\vec{w}}}) + (A\vec{v}_0,\vec{v}_0)$$

$$+ (\vec{\beta},\underline{\underline{\vec{w}}}) + (\vec{\beta},\vec{v}_0)$$

$$= \underbrace{(A\vec{w},\vec{w})}_{= s^2 - st + t^2} + (\overset{0}{\overset{\shortparallel}{\boxed{2A\vec{v}_0+\vec{\beta}}}},\vec{w}) + \underbrace{(A\vec{v}_0,\vec{v}_0) + (\vec{\beta},\vec{v}_0)}_{定数}$$

$$\left(\begin{pmatrix} *_1 \\ *_2 \end{pmatrix},\begin{pmatrix} s \\ t \end{pmatrix}\right) = *_1 s + *_2 t$$

$$= s^2 - st + t^2$$

a, b を選ぶ $\longleftrightarrow$ $\vec{v}_0 = \begin{pmatrix} a \\ b \end{pmatrix}$ を選ぶ.

$$A = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$$2A\vec{v}_0 + \vec{\beta} = \vec{0}$$

$$\longleftrightarrow \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \vec{0} \longleftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$z = \left(\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} s \\ t \end{pmatrix} \right) + \underbrace{\left(\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right) + \left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right)}_{= z(x=0,\ y=-2)}$$

$$\boxed{z = \left(\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) + \left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right)}$$

$$= x^2 - xy + y^2 - 2x + 4y$$

$$z = s^2 - st + t^2 - 4.$$

来週 別の計算を考える.

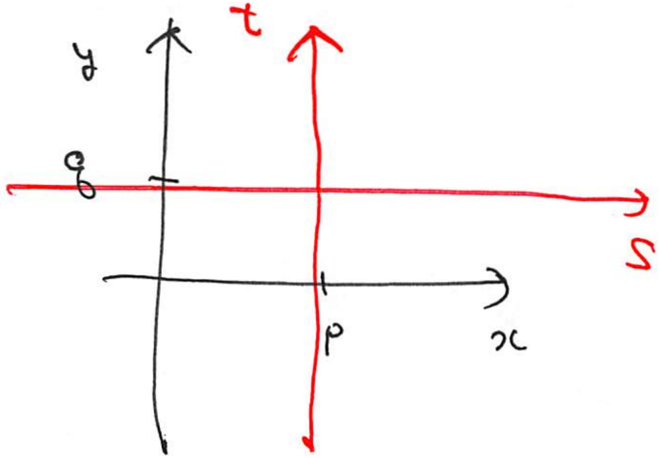
まとめ.

$$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad \vec{\beta} = \begin{pmatrix} d \\ e \end{pmatrix} \qquad |A| \neq 0$$

$$z = \left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) + \left(\vec{\beta}, \begin{pmatrix} x \\ y \end{pmatrix}\right) + f$$

$$= ax^2 + 2cxy + by^2 + dx + ey + f.$$

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}$$ とおく



$$z = \left(A\begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} s \\ t \end{pmatrix}\right) + f'$$

$$= as^2 + 2cst + bt^2 + f'$$

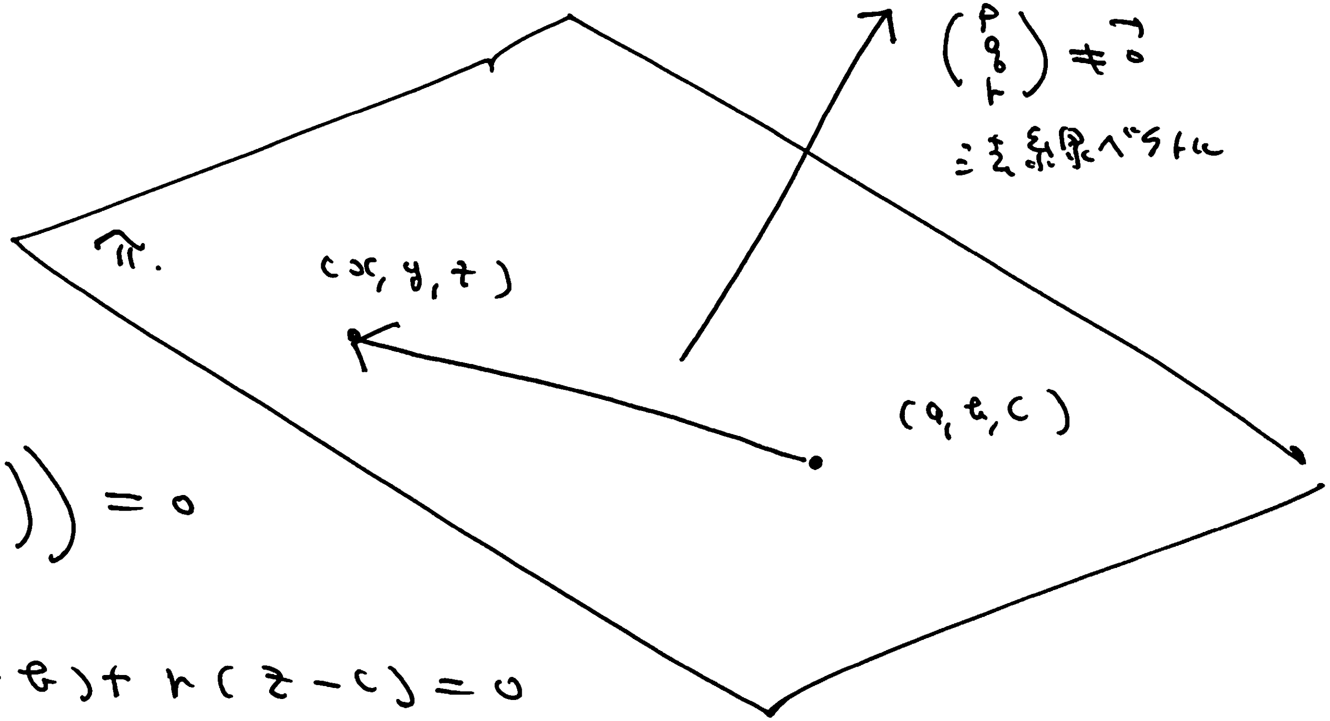
但し

$$2A\begin{pmatrix} p \\ q \end{pmatrix} + \vec{\beta} = \vec{0}$$

$$f' = f + \left(A\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix}\right) + \left(\vec{\beta}, \begin{pmatrix} p \\ q \end{pmatrix}\right)$$

$$= z(x = p, y = q)$$

平面 plane.



$$\left(\begin{pmatrix} x-a \\ y-b \\ z-c \end{pmatrix}, \begin{pmatrix} p \\ q \\ r \end{pmatrix} \right) = 0$$

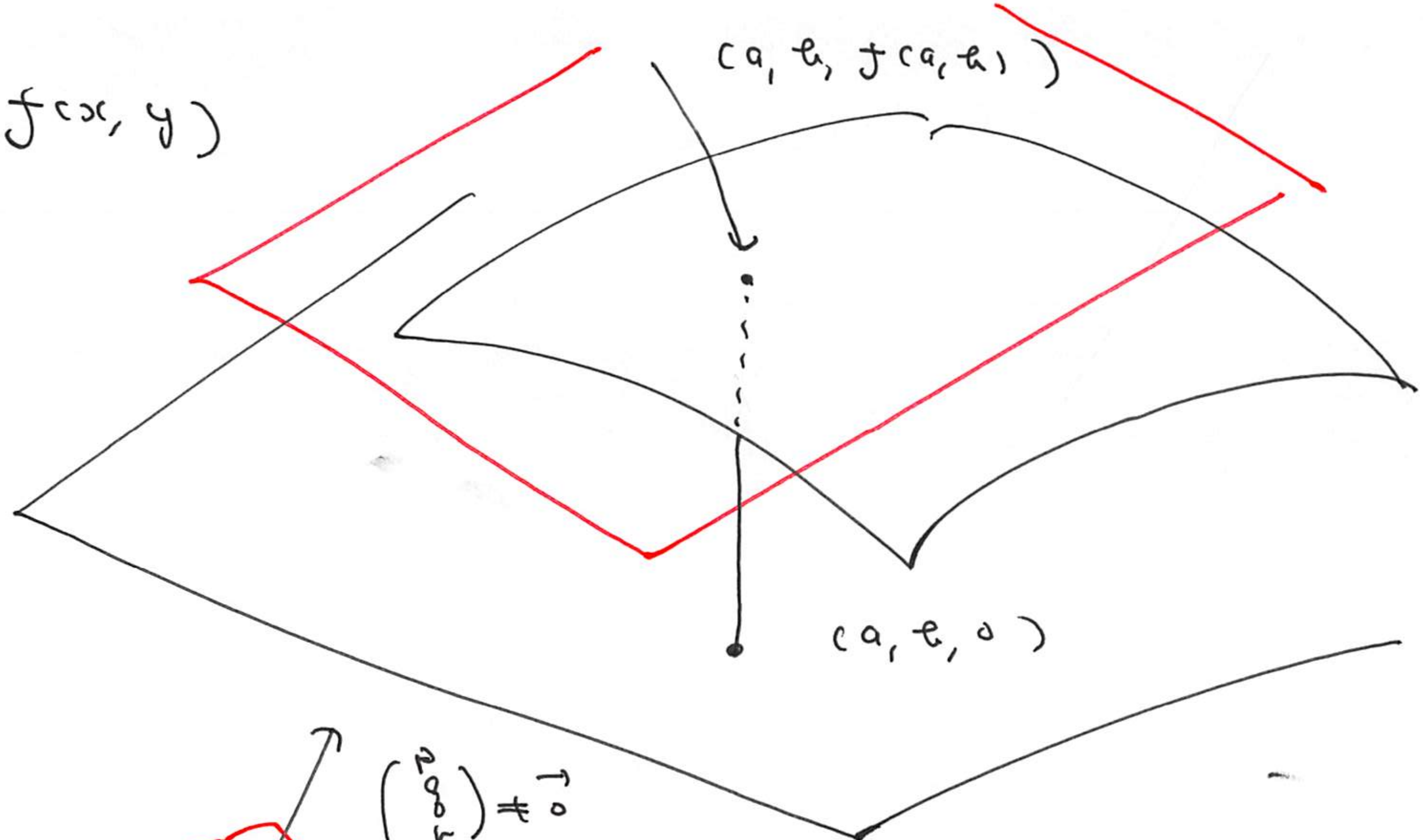
$$p(x-a) + q(y-b) + r(z-c) = 0$$

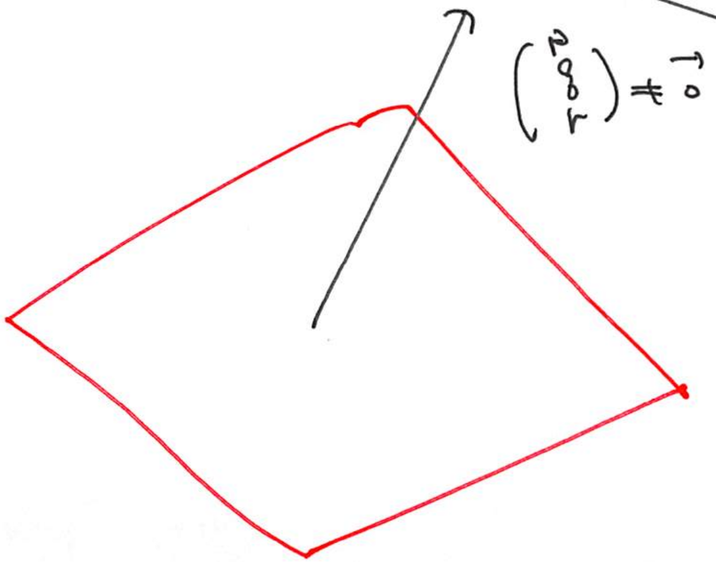
$r \neq 0$ のとき

$$z = -\frac{p}{r}(x-a) - \frac{q}{r}(y-b) + c$$

$$= A(x-a) + B(y-b) + c \quad \text{とおける.}$$

$z = f(x, y)$



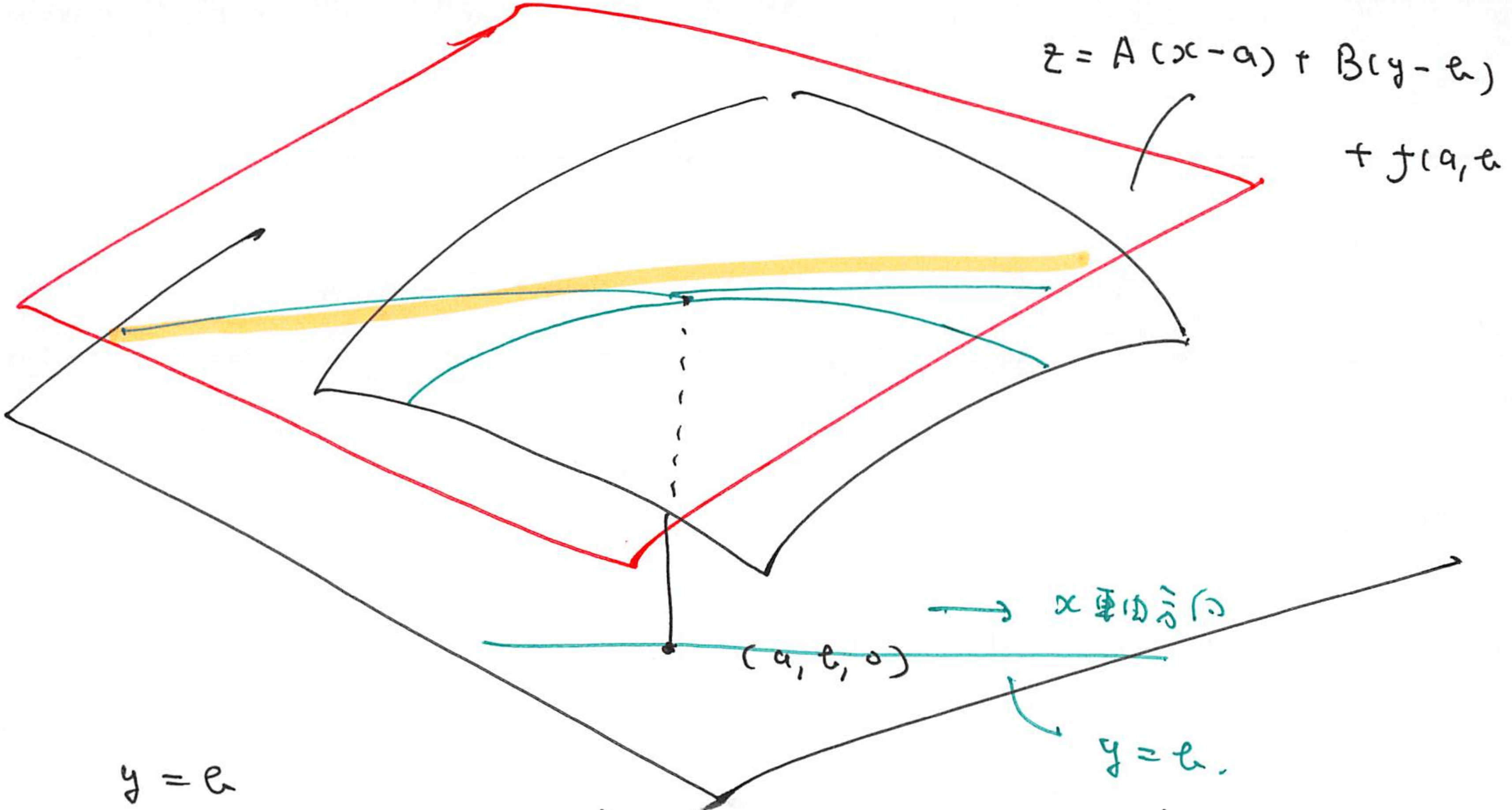


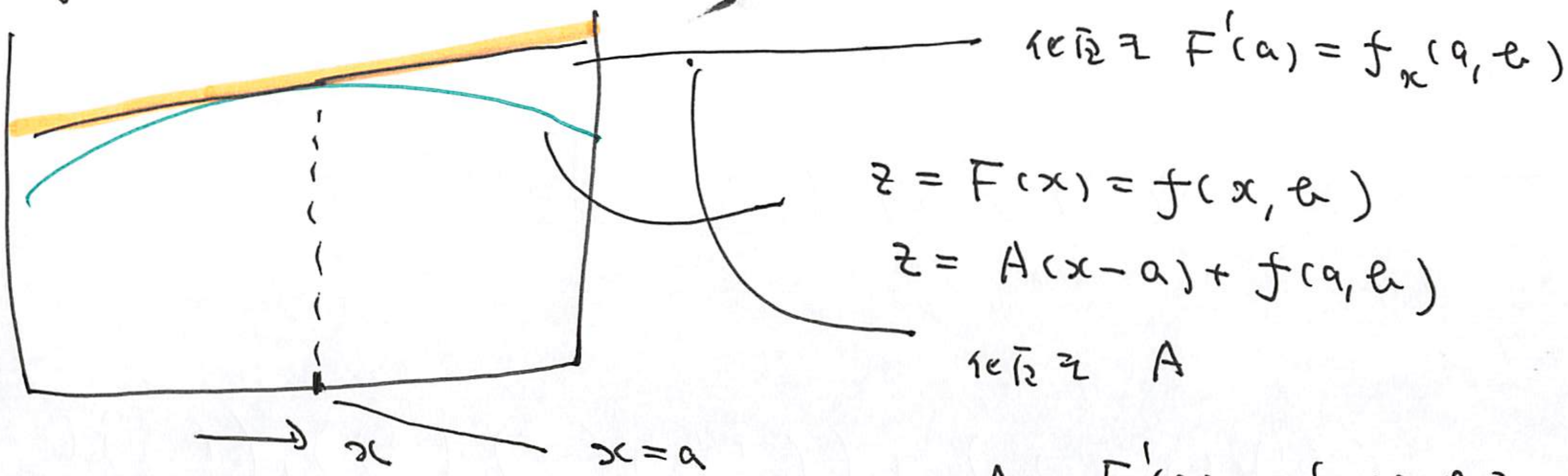
$r \neq 0$ に注意

$\begin{pmatrix} p \\ q \\ 0 \end{pmatrix} \parallel$ $x-y$平面

$\begin{pmatrix} p \\ q \\ 0 \end{pmatrix}$ に垂直な平面 $\perp$ $x-y$平面

$$z = A(x-a) + B(y-b) + f(a, b)$$





$$A = F'(a) = f_x(a, b)$$

同様に y 軸方向を考えると　$\beta = f_y(a, b)$

<u>まとめ</u>　$(a, b, f(a, b))$ における接平面

$$z = f_x(a, b)(x-a) + f_y(a, b)(y-b) + f(a, b)$$

例　$z = f(x, y) = x^2 - xy + y^2$ の　$(x, y, z) = (1, 1, 1)$

における接平面