I $A = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$ を回転の行列で対角化する.

極値を求める.

II. (1) $x^2 - xy + y^2 - 4x - y$

(2) $xy(2 - x - y)$

(3) $xy\, e^{x-y}$.

III (1) $x^2 + xy + y^2 = 1$ の下で

$$z = xy$$

IV (1) $A = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}$ (2) A^n を求める

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

を解く.

V $x^2 + y^2 + z^2 = 1$

$w = x + y - z$

$$A = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$$

$$|\lambda I_2 - A| = \begin{vmatrix} \lambda - 3 & +4 \\ +4 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda - 3)^2 - 4^2 = (\lambda - 7)(\lambda + 1)$$

固有値は $\lambda = 7$ と -1

$\underline{\lambda = 7}$

$$(\lambda I_2 - A)\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$$\begin{bmatrix} 4 & +4 \\ +4 & 4 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff x + y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\underline{\lambda = -1}$

$$\begin{pmatrix} -4 & +4 \\ +4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff x - y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ +x \end{pmatrix} = x\begin{pmatrix} 1 \\ +1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ +1 \end{pmatrix} = 0$$

$$\vec{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{p}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 \\ 1 \end{pmatrix}$$

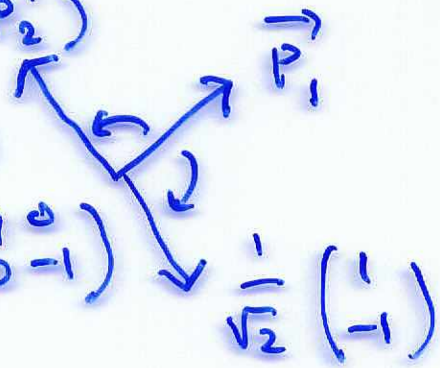
$$P = (\vec{p}_1\ \vec{p}_2) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$AP = (A\vec{p}_1\ \ A\vec{p}_2)$$

$$= (7\vec{p}_1\ \ -\vec{p}_2)$$

$$= (\vec{p}_1\ \vec{p}_2)\begin{pmatrix} 7 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= P\begin{pmatrix} 7 & 0 \\ 0 & -1 \end{pmatrix}$$



$$\left(A\binom{x}{y},\binom{x}{y}\right) = \left(\underbrace{AP\cdot{}^tP}_{=\,I_2}\binom{x}{y},\ P\,{}^tP\binom{x}{y}\right)$$

$$= 3x^2 - 8xy + 3y^2$$

$$= \left({}^tPAP\cdot{}^tP\binom{x}{y},\ {}^tP\binom{x}{y}\right)$$

$$P^{-1} = {}^tP.$$

$${}^tPP = P\,{}^tP = I_2$$

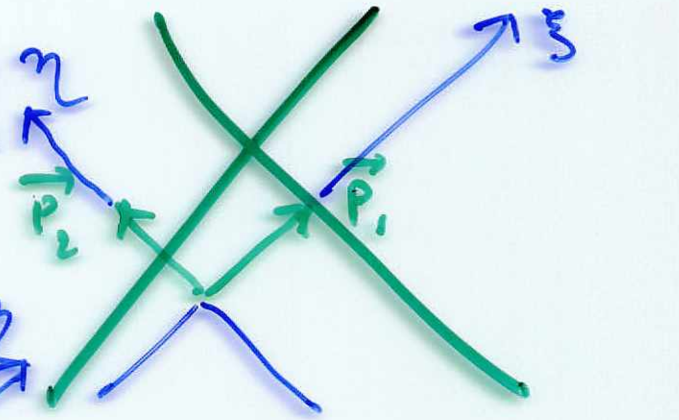
$$= \left(\begin{pmatrix}7&0\\0&-1\end{pmatrix}\binom{\xi}{\eta},\ \binom{\xi}{\eta}\right)$$

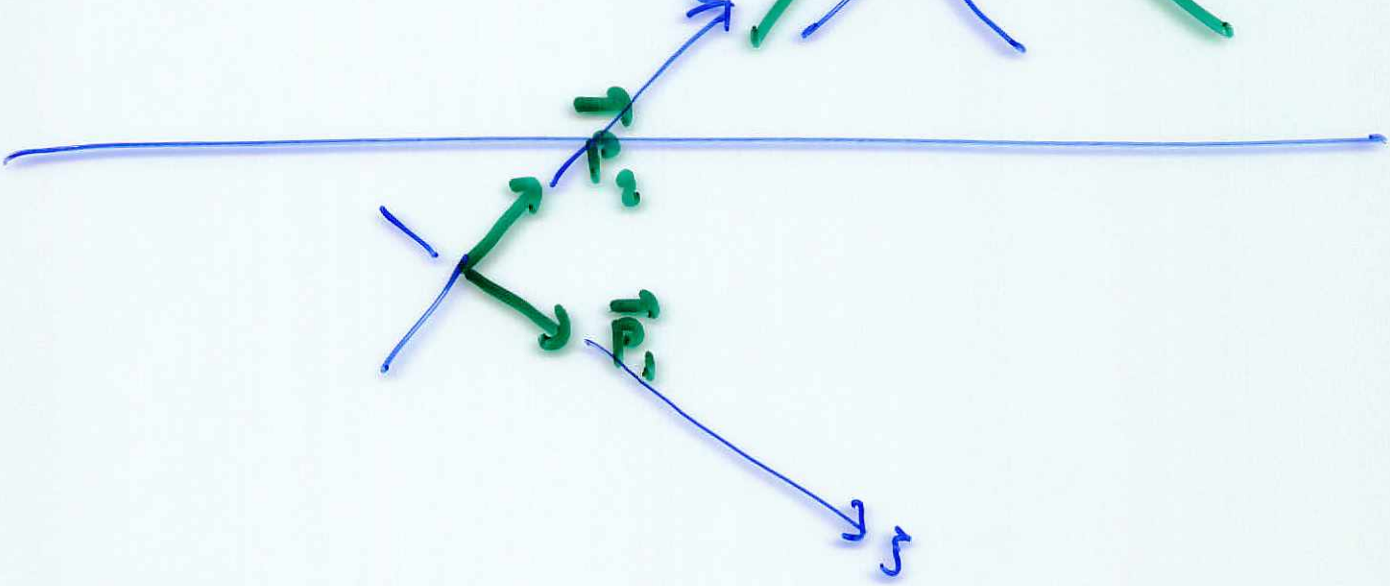
$$AP = P\begin{pmatrix}7&0\\0&-1\end{pmatrix}$$

$$= 7\xi^2 - \eta^2$$

$$\binom{\xi}{\eta} = {}^tP\binom{x}{y}$$

$$\binom{x}{y} = P\binom{\xi}{\eta} = \xi\vec{p}_1 + \eta\vec{p}_2$$





II (1) $z = x^2 - xy + y^2 - 4x - y$

$$\begin{cases} z_x = 2x - y - 4 = 0 \\ z_y = -x + 2y - 1 = 0 \end{cases}$$

$$(x, y) = (3, 2)$$

$z_{xx} = 2$, $z_{xy} = z_{yx} = -1$, $z_{yy} = 2$

$$\left.\begin{array}{l} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0 \\ z_{xx} = 2 > 0 \end{array}\right\} \to (3, 2) \text{ で極小.}$$

---

(2) $z = xy(2 - x - y)$ $\qquad (fg)' = f'g + fg'$

$z_x = y(2 - x - y) + xy \cdot (-1)$

$= y(2 - 2x - y) = 0$

$z_y = x(2 - x - 2y) = 0$

$$\boxed{a \cdot b = 0 \iff a = 0 \text{ OR } b = 0}$$

① $x = 0$, $y = 0$

② $y = 0$, $2 - x - 2y = 0 \rightsquigarrow x = 2, y = 0$

③ $2 - 2x - y = 0$, $x = 0 \rightsquigarrow x = 0$, $y = 2$

④ $2 - 2x - y = 0$, $2 - x - 2y = 0$, $x = y = \frac{2}{3}$

$z_{xx} = -2y$, $z_{yy} = -2x$.

$z_{xy} = (2 - 2x - y) + y \cdot (-1)$

$= 2 - 2x - 2y$.

$(0, 0)$

$\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = -4 < 0 \rightsquigarrow$ 極値をとらない.

$(0, 2)$

$\begin{vmatrix} -4 & -2 \\ -2 & 0 \end{vmatrix} = -4 < 0 \rightsquigarrow$ 同じ

$(2, 0)$

$\begin{vmatrix} 0 & -2 \\ -2 & -4 \end{vmatrix} = -4 < 0 \rightsquigarrow$ 同じ

$\left(\frac{2}{3}, \frac{2}{3}\right)$ $\begin{vmatrix} -\frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{4}{3} \end{vmatrix} = \left(\frac{4}{3}\right)^2 - \left(\frac{2}{3}\right)^2 > 0$

$z_{xx} = -\frac{4}{3} < 0$

$\rightarrow$ 極大

III

$x^2+xy+y^2=1$ の下で $z=xy$.

$g=x^2+xy+y^2-1$ $\qquad f=xy$.

$$\begin{cases} y=\lambda(2x+y) \cdots ① \\ x=\lambda(x+2y) \cdots ② \end{cases} \qquad \begin{cases} \nabla(f)=\lambda\nabla(g) \\ g=0 \end{cases}$$

$$\begin{pmatrix} 2\lambda & \lambda-1 \\ \lambda-1 & 2\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\vec{0}$$

$x=y=0$ は $g=0$ を満たさない.

$\begin{pmatrix} x \\ y \end{pmatrix} \neq \vec{0}$ とすると $\begin{vmatrix} 2\lambda & \lambda-1 \\ \lambda-1 & 2\lambda \end{vmatrix} = 4\lambda^2-(\lambda-1)^2=0$

$2\lambda=\lambda-1$ OR $2\lambda=-(\lambda-1)$

$\lambda=-1$, $\qquad \lambda=\frac{1}{3}$

$\lambda=-1$, $x+y=0$ $\quad y=-x$ を $g=0$ に代入

$x^2-x^2+x^2-1=0 \rightsquigarrow x=\pm1$

$(\pm1, \mp1)$

$\lambda=\frac{1}{3}$, $\quad \frac{2}{3}x-\frac{2}{3}y=0 \qquad x=y$,

$3x^2=1 \quad x=\pm\frac{\sqrt{3}}{3} \qquad \left(\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3}\right)$

① + ②

$x+y = 3\lambda(x+y) \longrightarrow \lambda = \frac{1}{3}$

OR

$x+y=0$

$$\begin{cases} y = \lambda(2x+y) & \text{--①} \\ x = \lambda(x+2y) & \text{--②} \end{cases}$$

① $\times (x+2y)$ − ② $\times (2x+y)$

$y(x+2y) - x(2x+y)$

$xy + 2y^2 - 2x^2 - xy = 0$

$\longrightarrow x^2 = y^2 \longrightarrow x = y$

$x = -y$

IV $A = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}$

A^n を 求める.

ケーリー・ハミルトン.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A^2 - (a+d)A + |A| \cdot I_2 = O_2$

$A^2 - 2A - 35 I_2 = O_2$

$\lambda^2 - 2\lambda - 35 = 0$ 固有方程式

$(\lambda - 7)(\lambda + 5)$

$A^2 - (7-5)A + 7 \cdot (-5) I_2 = O_2$

$$\begin{cases} (A^2 - 7A) = (-5)(A - 7I_2) \\ A^2 + 5A = 7(A + 5I_2) \end{cases}$$

$$\begin{cases} A(A - 7I_2) = (-5)(A - 7I_2) \\ A(A + 5I_2) = 7(A + 5I_2) \end{cases}$$

$$A^n(A - 7I_2) = (-5)^n (A - 7I_2)$$

$$-)\ A^n(A + 5I_2) = 7^n (A + 5I_2)$$

$$-12A^n = (-5)^n (A - 7I_2) - 7^n (A + 5I_2)$$

$$A^n = \frac{1}{12} 7^n (A + 5I_2) - \frac{1}{12} (-5)^n (A - 7I_2)$$

$$A = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}$$

$$|\lambda I_2 - A| = \begin{vmatrix} \lambda-1 & -12 \\ -3 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 36$$

$$= (\lambda-7)(\lambda+5)$$

$\underline{\lambda = 7}$ $\quad \begin{pmatrix} 6 & -12 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff x - 2y = 0$

固有ベクトルは

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad y \neq 0$$

$\underline{\lambda = -5}$ $\quad \begin{pmatrix} -6 & -12 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \iff x + 2y = 0$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (y \neq 0)$$

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$P = (\vec{v}_1 \ \vec{v}_2)$ とする.

$$AP = A(\vec{v}_1 \ \vec{v}_2) = (A\vec{v}_1 \ A\vec{v}_2)$$

$$= (7\vec{v}_1 \ -5\vec{v}_2)$$

$$= (\vec{v}_1 \ \vec{v}_2)\begin{pmatrix} 7 & 0 \\ 0 & -5 \end{pmatrix} = P\begin{pmatrix} 7 & 0 \\ 0 & -5 \end{pmatrix}$$

$|P| = \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = 2 + 2 = 4 \neq 0$ $\quad P$:正則.

定理 9.2 (309p.)

A : 2次正方行列 $\alpha \neq \beta$

$$\begin{cases} A\vec{v_1} = \alpha \vec{v_1}, & \vec{v_1} \neq \vec{0} \\ A\vec{v_2} = \beta \vec{v_2}, & \vec{v_2} \neq \vec{0} \end{cases}$$

$\Longrightarrow$ $P = (\vec{v_1}\ \vec{v_2})$ は正則

$P = (\vec{v_1}\ \vec{v_2})$ 正則

$\Longleftrightarrow \left(P\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Rightarrow x = y = 0 \right)$

$\Longleftrightarrow \left(x\vec{v_1} + y\vec{v_2} = \vec{0} \Rightarrow x = y = 0 \right)$

$x\vec{v_1} + y\vec{v_2} = \vec{0}$ ... (0)

$\boxed{\begin{aligned} &A(x\vec{v_1} + y\vec{v_2}) \\ &= xA\vec{v_1} + yA\vec{v_2} \end{aligned}}$

A・

$\alpha x\vec{v_1} + \beta y\vec{v_2} = \vec{0}$ ... (1)

$\beta\times$ $-)$ $\beta x\vec{v_1} + \beta y\vec{v_2} = \vec{0}$ ... (2)

$\underbrace{(\alpha - \beta)}_{\neq 0} x\vec{v_1} = \vec{0}$. $\rightarrow$ $x\vec{v_1} = \vec{0}$

$\downarrow$ $\vec{v_1} \neq \vec{0}$

$x = 0$

$y\vec{v_2} = \vec{0}$ $\xrightarrow[\vec{v_2} \neq \vec{0}]{}$ $y = 0$

$$A=\begin{pmatrix}1 & 12\\ 3 & 1\end{pmatrix}$$

$$\boxed{\frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}1 & 12\\ 3 & 1\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}}$$

$P=\begin{pmatrix}2 & -2\\ 1 & 1\end{pmatrix}$ にすると $AP=P\begin{pmatrix}7 & 0\\ 0 & -5\end{pmatrix}$

$$\left(P^{-1}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\right)'=P^{-1}\cdot\begin{pmatrix}x'(t)\\ y'(t)\end{pmatrix}$$

$$=P^{-1}A\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$$

$$=P^{-1}AP\cdot P^{-1}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$$

$$P^{-1}AP=\begin{pmatrix}7 & 0\\ 0 & -5\end{pmatrix}$$

$$=\begin{pmatrix}7 & 0\\ 0 & -5\end{pmatrix}P^{-1}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$$

$\begin{pmatrix}X(t)\\ Y(t)\end{pmatrix}=P^{-1}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$ とおく.

$$\frac{d}{dt}\begin{pmatrix}X(t)\\ Y(t)\end{pmatrix}=\begin{pmatrix}7 & 0\\ 0 & -5\end{pmatrix}\begin{pmatrix}X(t)\\ Y(t)\end{pmatrix}$$

$$X'(t)=7X(t),\quad Y'(t)=-5Y(t)$$

$$\boxed{f'(t)=\alpha f(t)\longrightarrow f(t)=f(0)e^{\alpha t}}$$

$$\begin{cases}X(t)=X(0)e^{7t}\\ Y(t)=Y(0)e^{-5t}\end{cases}$$

$$P^{-1}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} X(0)e^{7t} \\ Y(0)e^{-5t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{7t} & 0 \\ 0 & e^{-5t} \end{pmatrix} \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix}$$

$$= \begin{pmatrix} e^{7t} & 0 \\ 0 & e^{-5t} \end{pmatrix} P^{-1} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

$$\underset{P\cdot}{\rightsquigarrow} \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underbrace{P \begin{pmatrix} e^{7t} & 0 \\ 0 & e^{-5t} \end{pmatrix} P^{-1}}_{= e^{tA}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

$$e^{tA} := I_2 + tA + \frac{1}{2!} t^2 A^2 + \frac{1}{3!} t^3 A^3 + \cdots$$

$$e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots$$

$$A = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \text{ にチすいし}$$

$$AP = P \begin{pmatrix} 7 & 0 \\ 0 & -5 \end{pmatrix}$$

$$A = P \begin{pmatrix} 7 & 0 \\ 0 & -5 \end{pmatrix} P^{-1}$$

$$A^2 = P \begin{pmatrix} 7 & 0 \\ 0 & -5 \end{pmatrix} \overbrace{P^{-1} \cdot P}^{\substack{I_2 \\ \parallel}} \begin{pmatrix} 7 & 0 \\ 0 & -5 \end{pmatrix} P^{-1}$$

$$= P \begin{pmatrix} 7 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & -5 \end{pmatrix} P^{-1}$$

$$= P \begin{pmatrix} 7^2 & 0 \\ 0 & (-5)^2 \end{pmatrix} P^{-1}$$

$$A^n = P \begin{pmatrix} 7^n & 0 \\ 0 & (-5)^n \end{pmatrix} P^{-1}.$$

$$e^{tA} = \sum_{l=0}^{+\infty} \frac{1}{l!} t^l A^l$$

$$= \sum_{l=0}^{+\infty} \frac{t^l}{l!} P \begin{pmatrix} 7^l & 0 \\ 0 & (-5)^l \end{pmatrix} P^{-1}$$

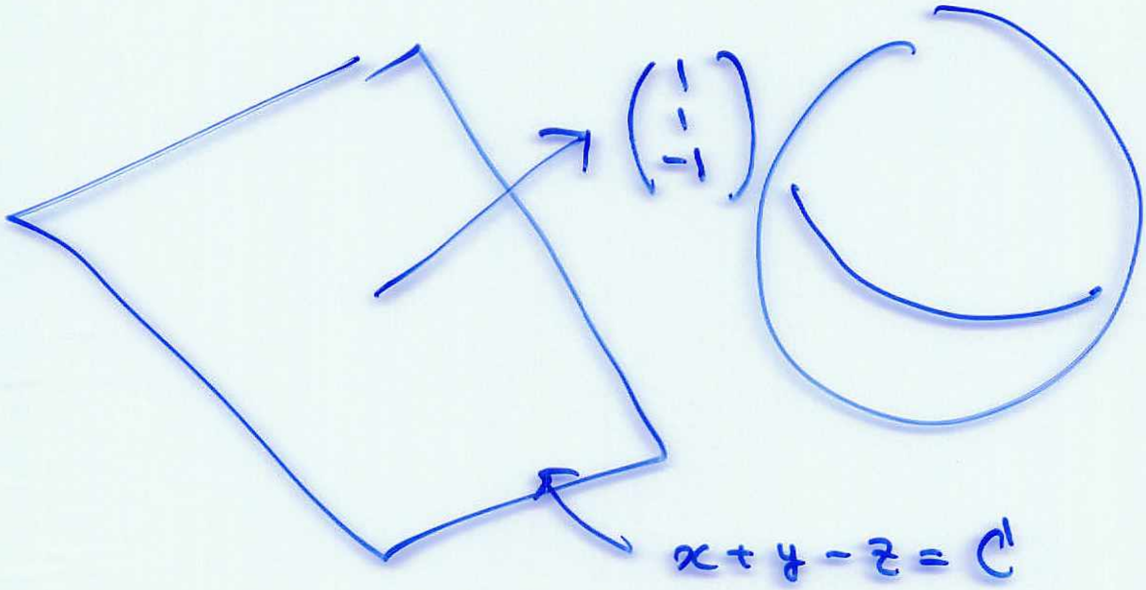
$\sum\limits^{+\infty}$ と P. は交換

$$= P \begin{pmatrix} \sum\limits_{l=0}^{+\infty} \frac{(7t)^l}{l!} & 0 \\ 0 & \sum\limits_{l=0}^{+\infty} \frac{(-5t)^l}{l!} \end{pmatrix} P^{-1}$$

$$= P \begin{pmatrix} e^{7t} & 0 \\ 0 & e^{-5t} \end{pmatrix} P^{-1}.$$

V $\quad x^2+y^2+z^2=1$ の下で

$$w = x+y-z.$$



$$
\begin{aligned}
1 &= \lambda \cdot 2x \\
1 &= \lambda \cdot 2y \qquad \lambda \neq 0 \\
-1 &= \lambda \cdot 2z
\end{aligned}
$$

$$x = \frac{1}{2\lambda},\quad y = \frac{1}{2\lambda},\quad z = -\frac{1}{2\lambda}$$

$x^2+y^2+z^2=1$ に代入

$$\frac{1}{4\lambda^2} \times 3 = 1 \qquad \lambda = \pm\frac{\sqrt{3}}{2}.\quad \frac{1}{\lambda} = \pm\frac{2}{\sqrt{3}}$$

$$(x, y, z) = \left(\pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}},\ \mp\frac{1}{\sqrt{3}}\right)$$

複号同順.