$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ ${}^tA = A$ 対称行列.

$\Phi_A(\lambda) = \det(\lambda I_2 - A)$

$= \lambda^2 - (a+b)\lambda + (ab - c^2)$

2実根を持つ

重根を持つ $\Longleftrightarrow A = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

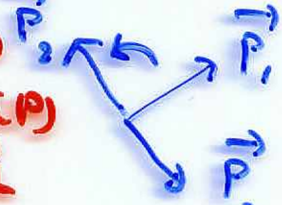
$\alpha \neq \beta$. 2根.

A：対称.

$A\vec{p}_1 = \alpha \vec{p}_1$, $A\vec{p}_2 = \beta \vec{p}_2$

$\|\vec{p}_1\| = \|\vec{p}_2\| = 1$, $\boxed{(\vec{p}_1, \vec{p}_2) = 0}$

$\vec{p}_1 \xrightarrow[90^\circ]{} \vec{p}_2$ ととる.



$\det(P) = 1$

$\det(P) = -1$

$P = (\vec{p}_1, \vec{p}_2)$ 回転

${}^tP = P^{-1}$

$AP = P\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

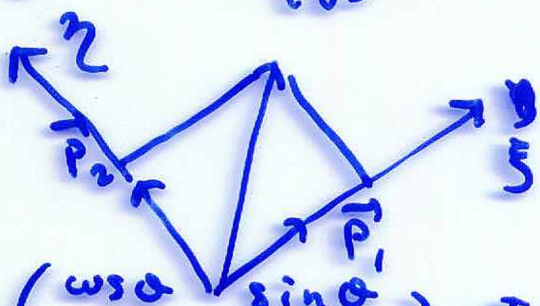
$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right)$

$= ax^2 + 2cxy + by^2$

$= \alpha\xi^2 + \beta\eta^2$

${}^tP\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = x\vec{p}_1 + y\vec{p}_2$



$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $P^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = {}^tP$

$P \cdot {}^tP = P \cdot P^{-1} = I_2$

P: 実 n次正方行列 $\quad P = (\vec{p}_1\ \vec{p}_2 \cdots \vec{p}_n)$

$$ {}^tPP = P\,{}^tP = I_n $$

$\rightsquigarrow$ 直交行列.

$$\Leftrightarrow \quad (\vec{p}_i, \vec{p}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\Leftrightarrow \quad (P\vec{x}, P\vec{y}) = (\vec{x}, \vec{y})$$

(全ての $\vec{x}, \vec{y}$)

$$ {}^tPP = I_n \rightsquigarrow \det({}^tPP) = \det(I_n) $$

$$ \det({}^tPP) = \det({}^tP)\det(P) = \det(P)^2, \qquad \det(I_n) = 1 $$

$$\boxed{\det({}^tP) = \det(P)}$$

$$\det(P) = \pm 1$$

$n = 2$. $\det(P) = -1$

は鏡映.

$$\left(A\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right) = \alpha\xi^2+\beta\eta^2 . \qquad A=\begin{pmatrix}a & c\\ c & b\end{pmatrix}$$

定理 $\left(A\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right) > 0$ 全ての $\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2$ $\neq\vec{0}$

$\iff \alpha,\beta > 0$

$\iff a>0,\ ab-c^2>0$

$$Q(x,y) = \left(A\begin{pmatrix}X\\Y\end{pmatrix},\begin{pmatrix}X\\Y\end{pmatrix}\right)$$

$$X = x-m_1,\quad Y = y-m_2$$

$$A = \frac{1}{1-\rho^2}\begin{pmatrix}\dfrac{1}{\sigma_1^2} & -\dfrac{\rho}{\sigma_1\sigma_2}\\[2ex] -\dfrac{\rho}{\sigma_1\sigma_2} & \dfrac{1}{\sigma_2^2}\end{pmatrix}$$

$|\rho|<1$

$$\frac{1}{1-\rho^2}\cdot\frac{1}{\sigma_1^2} > 0$$

$$\det(A) = \frac{1}{\sigma_1^2\sigma_2^2}\cdot\frac{1}{1-\rho^2} > 0$$

↑ 各自

$\left(A\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right)$：正定値とする.
$= Q(x,y)$

$$\iint_{\mathbb{R}^2} e^{-\frac{1}{2}Q(x,y)}\,dx\,dy.$$

$$= \iint_{\mathbb{R}^2} e^{-\frac{1}{2}(\alpha\xi^2+\beta\eta^2)}\,d\xi\,d\eta$$

$$\underset{\text{Fubini}}{=} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\alpha\xi^2}\,d\xi \cdot \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\beta\eta^2}\,d\eta$$

$$= \sqrt{2\pi}\cdot\frac{1}{\sqrt{\alpha}}\cdot\sqrt{2\pi}\cdot\frac{1}{\sqrt{\beta}} = 2\pi\frac{1}{\sqrt{\alpha\beta}} = 2\pi\frac{1}{\sqrt{\det A}}$$

$$\boxed{\alpha\beta = \det(A)}$$

$$(\lambda-\alpha)(\lambda-\beta) = \lambda^2-(a+b)\lambda+\det(A)$$

$$f(x,y) = \frac{1}{2\pi}\cdot\sqrt{\det A}\; e^{-\frac{1}{2}\left(\underbrace{A\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}}_{ax^2+2cxy+by^2}\right)}$$

確率密度, 但し $\left(A\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right)$ は正定値.

$$A = \begin{pmatrix}a & c\\ c & b\end{pmatrix}$$

$f(x, y)$ を用いて・

1) $E[X] = E[Y] = 0$ を示せ.

2) $E[X^2]$, $E[Y^2]$ を求めよ.

$V(X)$, $V(Y)$ を求めよ.

3) $E[XY]$ を求めよ.

ρ_{XY} を求めよ.

$$a x^2 + 2c\,xy + b y^2$$

$$= a\left(x + \frac{c}{a}y\right)^2 + b y^2 - \frac{c^2}{a} y^2$$

$$= a\left(x + \frac{c}{a}y\right)^2 + \boxed{\frac{ab - c^2}{a}}\, y^2$$

$= \gamma^2$ とおいて計算.

特性関数 $\theta \in \mathbb{R}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$ Eulerの公式

$$e^{t} = 1 + t + \frac{1}{2!}t^2 + \cdots + \frac{1}{k!}t^k + \cdots$$

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!} + \cdots + \frac{(-1)^k}{(2k)!}t^{2k} + \cdots$$

$$\sin t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots + \frac{(-1)^{k-1}}{(2k-1)!}t^{2k-1} + \cdots$$

$z \in \mathbb{C}$

$$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{1}{3!}z^3 + \cdots + \frac{1}{k!}z^k + \cdots$$

$\theta \in \mathbb{R}$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{1}{4!}\theta^4 + \cdots$$

$$= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - + \cdots\right)$$

$$+ i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

X: 確率変数　$\xi \in \mathbb{R}$

$$E[e^{i\xi X}] = \cancel{f(\xi)}\ \varphi(\xi)$$

X の特性関数

- $\sum_{n=0}^{+\infty} |a_n| < +\infty \Rightarrow \sum_{n=0}^{+\infty} a_n$ 収束.
- $\int_a^b |f(x)|\,dx < +\infty$

  $\Rightarrow \int_a^b f(x)\,dx$ 収束

$X \sim f(x)$ 確率密度

$$\int_{-\infty}^{+\infty} |e^{ix\xi} f(x)| = \int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$\rightsquigarrow \varphi(\xi) = \int_{-\infty}^{+\infty} e^{ix\xi} f(x)\,dx$$

全ての ξ で存在.

X パラメータ n, p の 2項変数

$$\varphi(\xi) = E[e^{i\xi X}]$$

$$= \sum_{k=0}^{n} p_k e^{i\xi k}$$

$$p_k = {}_nC_k \, p^k q^{n-k}$$

$$(q = 1 - p)$$

$$e^{i\xi k} = (e^{i\xi})^k$$