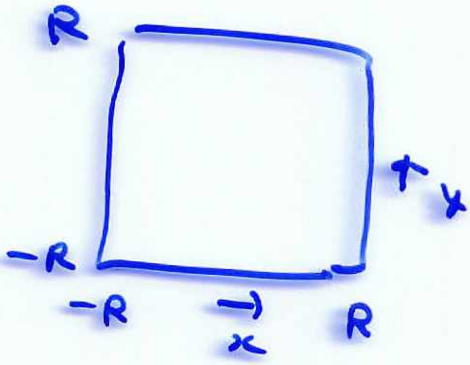
$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$$

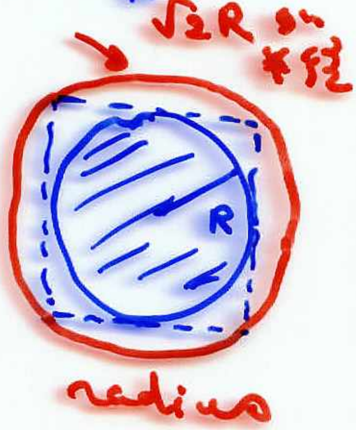
$$I_R := \int_{-R}^{R} e^{-\frac{x^2}{2}}\,dx$$

$$\iint_{[-R,R]\times[-R,R]} e^{-\frac{x^2}{2}}\cdot e^{-\frac{y^2}{2}}\,dx\,dy$$



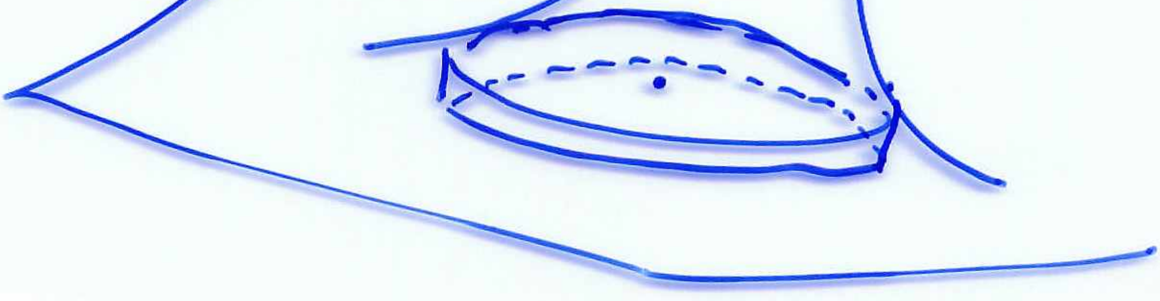
$$\underset{\substack{\uparrow\\ \text{Fubini}}}{=} \int_{-R}^{R} e^{-\frac{x^2}{2}}\,dx \cdot \int_{-R}^{R} e^{-\frac{y^2}{2}}\,dy = I_R^2$$

$$J_R = \iint_{\bigcirc R} e^{-\frac{x^2}{2}}\cdot e^{-\frac{y^2}{2}}\,dx\,dy$$



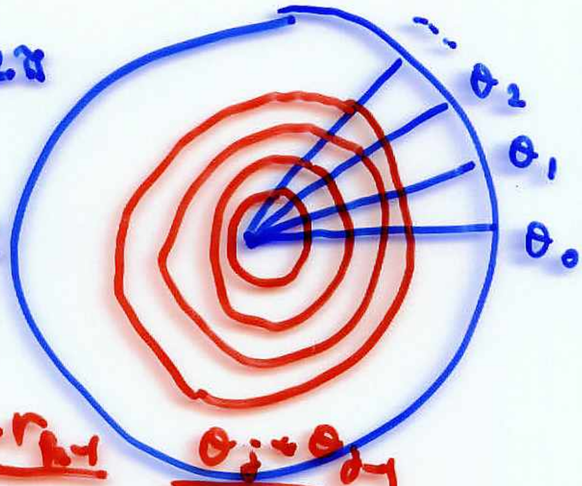
$$J_R \le I_R^2 \le J_{\sqrt{2}R}.$$

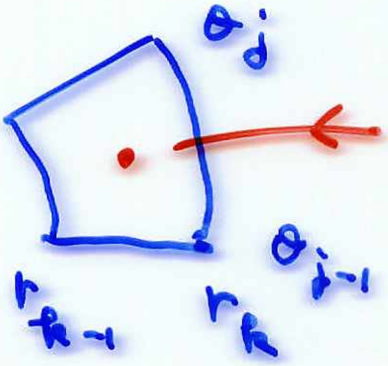
以下 J_R の $\lim_{R\to+\infty}$ を 求める.



$$0=\theta_0<\theta_1<\theta_2<\cdots<\theta_N=2\pi$$

$$0=r_0<r_1<r_2<\cdots<r_N=R$$





$\dfrac{r_k+r_{k-1}}{2}$, $\dfrac{\theta_j+\theta_{j-1}}{2}$

柱体.

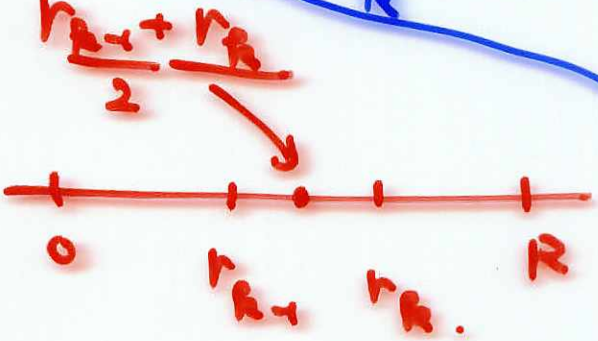
$$\text{面積}=\frac{1}{2}\left(r_k^2-r_{k-1}^2\right)\left(\theta_j-\theta_{j-1}\right)$$

$$\text{高さ}=e^{-\frac{1}{2}\left(\frac{r_k+r_{k-1}}{2}\right)^2}$$

$$J_R\fallingdotseq\sum_{j,k}e^{-\frac{1}{2}\left(\frac{r_k+r_{k-1}}{2}\right)^2}\frac{1}{2}\left(r_k^2-r_{k-1}^2\right)\times\left(\theta_j-\theta_{j-1}\right)$$

$$\sum_{j,k}\to\sum_k\sum_j$$

$$=2\pi\sum_k e^{-\frac{1}{2}\left(\frac{r_k+r_{k-1}}{2}\right)^2}\frac{1}{2}\left(r_k+r_{k-1}\right)\times\underbrace{\left(r_k-r_{k-1}\right)}$$



$[r_{k-1},\ r_k]$ の中点

$$e^{-\frac{1}{2}(x^2+y^2)} = e^{-\frac{1}{2}r^2}$$

$$=2\pi\int_0^R e^{-\frac{1}{2}r^2}\,r\,dr$$

$$\int_0^R e^{-\frac{r^2}{2}}\, r\, dr \qquad \left(e^{-\frac{1}{2}r^2}\right)' = -r\, e^{-\frac{1}{2}r^2}$$

$$= \left[-e^{-\frac{1}{2}r^2}\right]_0^R \qquad R \to +\infty$$

$$= \left(1 - e^{-\frac{1}{2}R^2}\right) \longrightarrow 1$$

$$J_R \to 2\pi \quad (R \to +\infty)$$

$$J_{\sqrt{2}R} \longrightarrow 2\pi$$

$$\begin{array}{ccccc} J_R & \leq & I_R^2 & \leq & J_{\sqrt{2}R} \\ \downarrow & & & & \downarrow \\ 2\pi & & & & 2\pi \end{array}$$

$$I_R^2 \longrightarrow 2\pi \qquad \left(\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx\right)^2$$

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

$$\iint f(x,y)\, dx\, dy$$

$$= \int_0^R \left(\int_{\theta_0}^{\theta_1} f(r\cos\theta, r\sin\theta)\, d\theta\right) r\, dr$$

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X と Y は独立

$X \sim N_{m_1, \sigma_1}$ 　期待値 m_1, 分散 σ_1) の正規分布

$Y \sim N_{m_2, \sigma_2}$

$$\Rightarrow \; Z = X + Y \sim N_{m_1 + m_2, \sigma_3}$$

但し $\sigma_3^2 = \sigma_1^2 + \sigma_2^2$

後に特性関数を使って示す.

復習

$$\left.\begin{array}{l} X \sim f(x) \\ Y \sim g(y) \end{array}\right\} \text{独立} \Rightarrow Z = X + Y \sim h(z)$$

$$h(z) = \int_{-\infty}^{+\infty} f(z-y)\, g(y)\, dy.$$

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} f(x)\, dx.$$

$$= E\left[e^{-iX\xi}\right]$$

X の特性関数.

中心極限定理を解決するときに使う.

$\alpha > 0$

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx.$$

ガンマ関数

$$\Gamma(1) = \int_0^{+\infty} e^{-x} dx = \Big[-e^{-x}\Big]_0^{+\infty} = 1$$

$$\Gamma(\tfrac{1}{2}) = \int_{-\infty}^{+\infty} x^{-\frac{1}{2}} e^{-x} dx = ?$$

$$\int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} dt = \sqrt{2\pi} \qquad x = \frac{t^2}{2}$$

$$= 2\int_0^{+\infty} e^{-\frac{t^2}{2}} dt \qquad dx = t\,dt$$

$$dt = \frac{1}{t} dx$$

$$= 2\int_0^{+\infty} e^{-x} \frac{1}{\sqrt{2x}} dx \qquad = \frac{1}{\sqrt{2x}} dx$$

$$= \sqrt{2} \int_{-\infty}^{+\infty} x^{-\frac{1}{2}} e^{-x} dx$$

$$= \sqrt{2}\, \Gamma(\tfrac{1}{2}) \rightsquigarrow \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(s+1) = \int_0^{+\infty} x^s e^{-x} \, dx$$

$$= \int_0^{+\infty} x^s \left(-e^{-x}\right)' dx$$

$$= \left[-x^s e^{-x}\right]_0^{+\infty} + s\int_0^{+\infty} x^{s-1} e^{-x} dx$$

$$= s\,\Gamma(s)$$

'公式' $\quad \Gamma(s+1) = s\,\Gamma(s)$

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1) = \cdots$$

$$= n(n-1)\cdots 2\cdot 1\,\Gamma(1) \qquad \Gamma(2) = 1\cdot\Gamma(1)$$

$$= n!$$

'公式' $\quad \Gamma(n+1) = n!$

$X \sim N_{0,1}$

$$E[X^{2k}] = \int_{-\infty}^{+\infty} \frac{x^{2k}}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 2\int_0^{+\infty} x^{2k} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \qquad t = \frac{x^2}{2}$$

$$= \frac{2^k}{\sqrt{\pi}} \int_0^{+\infty} t^{k+\frac{1}{2}-1} e^{-t} dt$$

$$= \frac{2^k}{\sqrt{\pi}} \Gamma\left(k+\frac{1}{2}\right)$$

问 $\quad E[X^6] = ?, \quad E[X^8] = ?$