標準正規分布

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$$

（後期にて
2通り説明

$$\left.\begin{aligned} N_{0,1}(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \geqq 0 \\ \int_{-\infty}^{+\infty} N_{0,1}(x)\,dx &= 1 \end{aligned}\right\}$$

$N_{0,1}(x)$: 標準正規分布の
密度関数.

$$X \sim N_{0,1}(x)$$

$$E(X) = \int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \lim_{R\to+\infty} \overbrace{\int_{-R}^{R} \frac{1}{\sqrt{2\pi}} f(x)\,dx}^{=0} = 0$$

一般に

$$\left.\begin{aligned} f(x) &= x\, e^{-\frac{x^2}{2}} \text{ とおくと} \\ f(-x) &= -f(x) \end{aligned}\right\} \text{奇関数.}$$

$$g: \text{奇関数} \quad \underbrace{\int_{-R}^{R} g(x)\,dx}_{=} = 0$$

$$\underbrace{\int_{-R}^{0} g(x)\,dx}_{} + \int_{0}^{R} g(x)\,dx.$$

$$(x = -y \qquad dx = -dy$$

$$= \int_{R}^{0} g(-y)\cdot(-dy)$$

$$= \int_{0}^{R} (-1)\, g(y)\,dy$$

$$= -\int_{0}^{R} g(y)\,dy.$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 \, N_{0,1}(x)\, dx$$

$$= \int_{-\infty}^{+\infty} \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$\boxed{\left(e^{-\frac{x^2}{2}}\right)' = e^{-\frac{x^2}{2}} \cdot (-x)}$$

$$= \int_{-\infty}^{+\infty} \frac{(-x)}{\sqrt{2\pi}} \left(e^{-\frac{x^2}{2}}\right)' dx$$

$$= \left[\frac{(-x)}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right]_{-\infty}^{+\infty} + \int^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx .$$

$\frac{x^2}{2} = y \longrightarrow +\infty$ $\quad x e^{-\frac{x^2}{2}}$

$(x \to \pm\infty)$ $\quad = \frac{\pm\sqrt{y}}{e^{+y}} \longrightarrow 0$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = 1$$

$$V(X) = E((X-\mu)^2) \quad \mu = E(X)$$

$$= E(X^2) - (E(X))^2$$

$$V(X) = 1 - 0^2 = 1$$

<u>まとめ</u> $X \sim N_{0,1}$ のとき

$E(X) = 0$, $V(X) = 1$

一般の正規分布

$$N_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

$\sigma > 0$

$m \in \mathbb{R}$

$$z = \frac{x-m}{\sigma} \qquad dz = \frac{dx}{\sigma}$$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}} dz = 1$$

$$X \sim N_{m,\sigma}(x) \Rightarrow Z = \frac{X-m}{\sigma} \sim N_{0,1}(z)$$

$$P(a \le \underset{\substack{\parallel \\ \frac{Z-m}{\sigma}}}{X} \le b) = \underbrace{P(m - \sigma a \le Z \le m + \sigma b)}_{\uparrow}$$

標準正規分布表で計算できる.

$$X \sim N_{m,\sigma}(x)$$

($E(X^k)$については
非常重要計算する)

$$E(X) = \int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

$$z = \frac{x-m}{\sigma}$$

$$= \int_{-\infty}^{+\infty} \boxed{\frac{(x-m)}{\sqrt{2\pi}} \cdot \frac{1}{\sigma}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

$$+ \int_{-\infty}^{+\infty} m \underbrace{\boxed{\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} dx}}_{\text{密度関数}}$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \underbrace{z\, e^{-\frac{z^2}{2}}}_{\text{奇関数}} \cdot \sigma dz + m$$

$$= 0 + m = m$$

$$E(X^2) = ? \qquad z = \frac{x-m}{\sigma} \quad dz = \frac{dx}{\sigma}$$

$$V(X) = \int_{-\infty}^{+\infty} (x-m)^2 \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{+\infty} (\sigma z)^2 \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}} dz$$

$$= \sigma^2 \underbrace{\int_{-\infty}^{+\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz}_{\shortparallel}$$

zは標準正規分布.

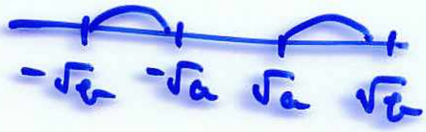
$$E(Z^2) = 1$$

$$= \sigma^2.$$

$X \sim N_{0,1}$ 　χ^2分布について

$Y = X^2$. Y の密度関数.

$P(a \leqq Y \leqq b)$ 　$a, b \geqq 0$ 　$a < b$ とする



$= P(-\sqrt{b} \leqq X \leqq -\sqrt{a}$ OR $\sqrt{a} \leqq X \leqq \sqrt{b})$

$$= \underbrace{\int_{-\sqrt{b}}^{-\sqrt{a}} N_{0,1}(x)\,dx}_{= \int_{\sqrt{a}}^{\sqrt{b}} N_{0,1}(t)\,dt \quad (x=-t)} + \int_{\sqrt{a}}^{\sqrt{b}} N_{0,1}(x)\,dx$$

$\boxed{N_{0,1}(x) \text{ は even}}$

$$= 2\int_{\sqrt{a}}^{\sqrt{b}} N_{0,1}(x)\,dx$$

$x^2 = y$ によって変数変換

$$= 2\int_a^b \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}}\,dy$$

$2x\,dx = dy$

$dx = \frac{1}{2x}\,dy$

$= \frac{1}{2\sqrt{y}}\,dy$

$$= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \cdot \frac{1}{\sqrt{y}}\,dy.$$

$$Y \sim \begin{cases} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{y}} e^{-\frac{y}{2}} & (y \geqq 0) \\ 0 & (y \leqq 0) \end{cases}$$

自由度0の χ^2 分布.

$$\int_0^1 \frac{1}{\sqrt{y}}\,dy$$

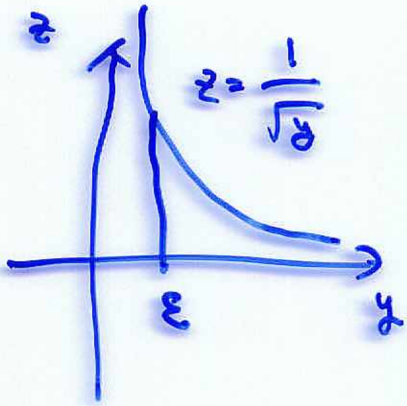
$$\shortparallel$$

$$\lim_{\varepsilon \to +0} \left[\int_\varepsilon^1 \frac{1}{\sqrt{y}}\,dy \right]$$

$$= \lim_{\varepsilon \to +0} \left[2\sqrt{y} \right]_\varepsilon^1$$

$$= \lim_{\varepsilon \to +0} \left[2\sqrt{1} - 2\sqrt{\varepsilon} \right]$$

$$= 2.$$



$y \to +0$ のとき

$$\frac{1}{\sqrt{y}} \to +\infty$$

$$(\sqrt{y})' = \frac{1}{2}\frac{1}{\sqrt{y}}$$

$$(2\sqrt{y})' = \frac{1}{\sqrt{y}}$$

復習 $X \sim f(x),\ Y \sim g(y)$

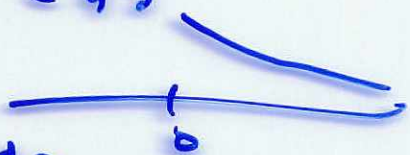
X と Y 独立.

$Z = X + Y \sim h(z)$

$$h(z) = \int_{-\infty}^{+\infty} f(x)\, g(z-x)\, dx .$$

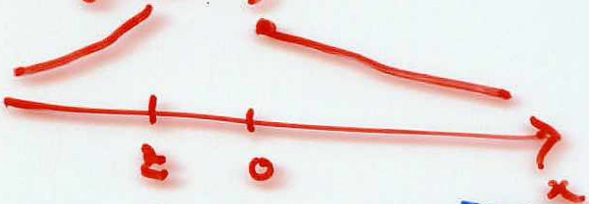
特に

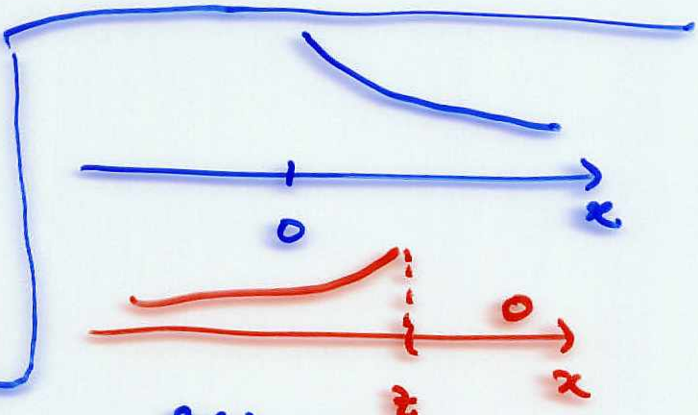
$$\begin{cases} f(x) = 0 & x < 0 \\ g(y) = 0 & y < 0 \end{cases}$$ とする



$$h(z) = \int_{-\infty}^{+\infty} f(x)\, \underline{g(z-x)}\, dx$$

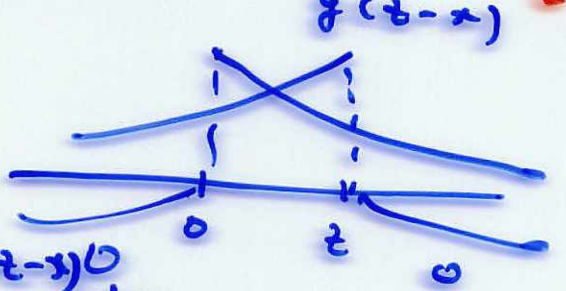
- $z < 0$ のとき





$h(z) = 0$

- $z \geq 0$



$$h(z) = \int_0^z f(x)\, g(z-x)\, dx$$

$X_1, X_2 \sim N_{0,1}$ 独立

$Y_1 = X_1^2$, $\cancel{X}_2$ $Y_2 = X_2^2 \rightsquigarrow$ 独立. ⟨ gap 存

$Z = Y_1 + Y_2$

$$Y_1 \sim \begin{cases} \dfrac{1}{\sqrt{2\pi}} \dfrac{1}{\sqrt{y_1}} e^{-\frac{y_1}{2}} & (y_1 \geq 0) \\ 0 & (\text{otherwise}) \end{cases}$$

$z \geq 0$

$$h(z) = \int_0^z \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{y_1}} e^{-\frac{y_1}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{z-y_1}} e^{-\frac{(z-y_1)}{2}} dy_1$$

$$= \frac{e^{-\frac{z}{2}}}{2\pi} \int_0^z \frac{1}{\sqrt{y_1(z-y_1)}} dy_1$$

$$= \frac{e^{-z}}{2\pi} \cdot \pi = \frac{1}{2} e^{-\frac{z}{2}}.$$

$z \to a$

$$\int_0^a \frac{1}{\sqrt{x(a-x)}}\,dx$$

$$\begin{aligned} x(a-x) &= ax - x^2 \\ &= -(x^2 - ax) \\ &= -\left(x-\frac{a}{2}\right)^2 + \frac{a^2}{4} \end{aligned}$$

$$= \int_0^a \frac{dx}{\sqrt{\left(\frac{a}{2}\right)^2 - \left(x - \left(\frac{a}{2}\right)\right)^2}}$$

$$= \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{dt}{\sqrt{\left(\frac{a}{2}\right)^2 - t^2}} \qquad t = x - \frac{a}{2}, \quad dt = dx$$

tについて偶関数.

$$= 2\int_0^{\frac{a}{2}} \frac{dt}{\sqrt{\left(\frac{a}{2}\right)^2 - t^2}} \qquad \boxed{t = \frac{a}{2}\sin\theta}$$

$$dt = \frac{a}{2}\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{\frac{a}{2}\cos\theta\, d\theta}{\frac{a}{2}\cos\theta}$$

$$\begin{aligned} \sqrt{} &= \frac{a}{2}\sqrt{1-\sin^2\theta} \\ &= \frac{a}{2}\sqrt{\cos^2\theta} \\ &= \frac{a}{2}\cos\theta \end{aligned}$$

$$\left(0 \le \theta \le \frac{\pi}{2}\right)$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta = 2 \times \frac{\pi}{2} = \pi$$

$x_1, x_2 \sim N_{0,1}$ 独立.

$$z = x_1^2 + x_2^2$$

$$\sim \begin{cases} \cancel{\frac{1}{2\pi}}\, e^{-\frac{z}{2}} & (z > 0) \\ 0 & (z \le 0) \end{cases}$$

仮

$$U = \frac{X_1^2 + X_2^2}{2} = \frac{Z}{2}$$ 自由度1のχ^2分布.

$\alpha > 0$ $\quad X \sim f(x)$

$$U = \frac{X}{\alpha} \qquad u = \frac{x}{\alpha} \quad du = \frac{dx}{\alpha}$$

$$P(a \leq U \leq b) = P(\alpha a \leq X \leq \alpha b)$$

$$= \int_{\alpha a}^{\alpha b} f(x)\,dx$$

$$= \int_a^b f(\alpha u) \cdot \alpha\,du$$

$$U \sim f(\alpha u) \cdot \alpha$$

$$U = \frac{X_1^2 + X_2^2}{2} \sim \frac{1}{\pi} e^{-u}$$

自由度nのχ^2分布については

後期において説明.

まとめ

$$X \sim f(x),\ \alpha > 0 \Rightarrow U = \frac{X}{\alpha} \sim f(\alpha u) \cdot \alpha$$