$$\mathbb{Z}_+^2 = \{(i,j);\ i,j \in \mathbb{Z}_+\}$$

$$\mathbb{Z}_+ = \{0,1,2,\cdots\} \xrightarrow{f} \mathbb{R}$$

$(X,Y) \in \mathbb{Z}_+^2$ である確率変数

$$P(X=i,\ Y=j) = p_{ij}$$

$$\sum_{i,j} p_{ij} = 1,\quad p_{ij} \geqq 0$$

$$V(f(X,Y)) = \sum_{i,j} \left(f(i,j) - E(f(X,Y))\right)^2 \times p_{ij}$$

$$V(X) = \sum_{i=0}^{+\infty} (i - E(X))^2 p_i$$

$$p_i = \sum_{i,j} p_{ij}$$

$$= \sum_{i,j} (i - E(X))^2 p_{ij}$$

$\sum_{i,j} |\quad| p_{ij} < +\infty$ の下で

$$= \sum_i \sum_j \underbrace{(i - E(X))^2}_{j\text{に依らない}} p_{ij}$$

$$= \sum_i (i - E(X))^2 \sum_j p_{ij}$$

$$= \sum_i (i - E(X))^2 p_i$$

$$C(x, y) = \sum_{i,j} (i - \lambda)(j - \mu) p_{ij}$$ 共分散.

$$\lambda = E(x),\ \mu = E(y) \quad \Updownarrow$$

$$C_{xy} = \sum_i (x_i - E(x))(y_i - E(y))$$

$$= \sum_{i,j} ij\, p_{ij} - \lambda \overset{= E(y)}{\boxed{\sum_{i,j} j\, p_{ij}}} - \mu \overset{= E(x)}{\boxed{\sum_{i,j} i\, p_{ij}}} + \lambda\mu \overset{= 1}{\boxed{\sum_{i,j} p_{ij}}}$$

$$= \sum_i \sum_j i\, p_{ij}$$

$$= \sum_i i \sum_j p_{ij} = \sum_i i\, p_i$$

$$= E(xy) - E(x)E(y).$$

まとめ $E(x, y) = E(xy) - E(x)E(y)$

---

x と y が独立.

$$\Longleftrightarrow P(X = i,\ Y = j) = P(X = i)\ P(Y = j)$$

$p_{ij} = p_i \cdot p'_j$ となる.

---

x, y が独立とすると

$$E(xy) = E(x)\,E(y).$$

このとき $C(x, y) = 0$ (x と y は無相関)

$$E(xy) = \sum_{i,j} ij\, p_{ij}$$

$$= \sum_j \sum_i ij\, p_i p'_j$$

$$= \sum_j j p'_j \left(\sum_i i\, p_i \right) \to (*)$$

$$\left(p_i = \sum_j p_{ij} ,\quad p'_j = \sum_i p_{ij} \right)$$

$$\underset{(*)}{E(xy)} = \sum_j j p'_j \cdot \underbrace{E(x)}_{j\text{ に依らない...}}$$

$$= E(y) E(x)$$

$$V(x+y) = \sum_{i,j} \Big((i+j) - \underbrace{E(x+y)}_{\overset{\shortparallel}{\lambda+\mu}} \Big)^2 p_{ij}$$

$$= \sum_{i,j} \left[(i-\lambda) + (j-\mu) \right]^2 p_{ij}$$

$$= \sum_{i,j} (i-\lambda)^2 p_{ij} + 2 \sum_{i,j} (i-\lambda)(j-\mu) p_{ij} + \sum_{i,j} (j-\mu)^2 p_{ij}$$

$$= V(x) + 2C(x,y) + V(y)$$

まとめ $V(x+y) = V(x) + 2\overbrace{C(x,y)}^{\overset{0}{\shortparallel}} + V(y)$

X と Y が独立 $\Longrightarrow$ $V(x+y) = V(x) + V(y)$

$$\mathbb{Z}_+^n = \mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+ \ni (x_1, x_2, \cdots, x_n)$$

$$P(x_1 = j_1, x_2 = j_2, \cdots, x_n = j_n) = P_{j_1 j_2 \cdots j_n}$$

$$\sum P_{j_1 j_2 \cdots j_n} = 1, \quad P_{j_1 \cdots j_n} \geqq 0$$

$x_1, \cdots, x_n$ が独立

$$\Leftrightarrow P(x_1 = j_1, \cdots, x_n = j_n) = P(x_1 = j_1) \cdot \cdots \cdot P(x_n = j_n)$$

独立のとき

$$V(x_1 + \cdots + x_n) = V(x_1) + \cdots + V(x_n)$$

(← 一般的に

$$\sum_{j=1}^{n} V(x_j) + 2\sum_{\substack{i,j \\ i \neq j}} C(x_i, x_j)$$

$\overset{\shortparallel}{0}$ $x_1, \cdots, x_n$ 独立のとき.

モデル

$X \quad \mathbb{Z}_+ , \{p_j\}$

n sample 標本.

$(X_1, \cdots, X_n)$

独立

$$P(X_1 = j_1, \cdots, X_n = j_n) = p_{j_1} p_{j_2} \cdots p_{j_n}$$

$E(X)$ 母平均.

$\updownarrow$

Y 標本平均.

$$Y = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$E(Y) = \frac{1}{n}\left(E(X_1) + \cdots + E(X_n)\right) = E(X)$$

$$V(Y) = V\left(\frac{1}{n}(X_1 + \cdots + X_n)\right) = \frac{1}{n^2} V(X_1 + \cdots + X_n) = \frac{1}{n^2}\left(V(X_1) + \cdots + V(X_n)\right) = \frac{1}{n^2} \cdot n V(X) = \frac{1}{n} V(X)$$

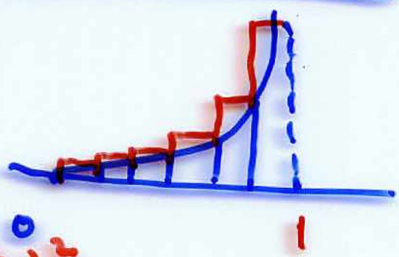
積分とは　$f : [a, b] \longrightarrow \mathbb{R}$

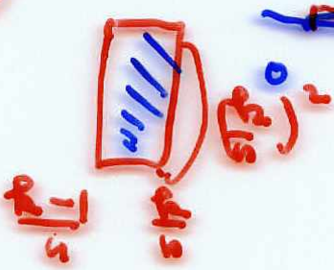
連続.

$$\int_a^b f(x)\,dx = ?$$

$$\boxed{\begin{array}{l} \displaystyle\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \\ \displaystyle\left(\frac{x^3}{3}\right)' = x^2 \end{array}}$$

$$\sum_{k=1}^{n} \frac{1}{n}\cdot\left(\frac{k}{n}\right)^2$$



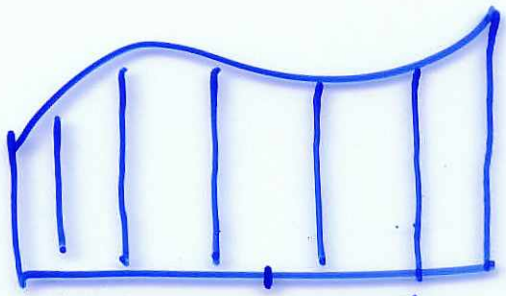


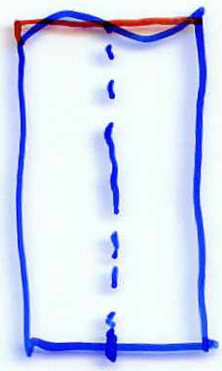
$$= \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$n \to +\infty$

$$= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \longrightarrow \frac{1}{6}\cdot 1\cdot 2 = \frac{1}{3}$$





$x_0 = a$, x_1, x_2, x_{N-1}, $b = x_N$　　x_{i-1}, ξ_i, x_i

Δ: 分割

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

$$\sum_{i=1}^{N} f(\xi_i)(x_i - x_{i-1}) \longrightarrow \int_a^b f(x)\,dx.$$

$$|\Delta| = \max(x_i - x_{i-1}) \qquad |\Delta| \to 0$$

分割の幅

<u>定理</u> 微分積分学の基本定理.

$$F(x) = \int_a^x f(t)\,dt$$

(仮) f: $[a, b]$ 上連続.

このとき F は微分可能

$$F'(x) = f(x)$$

$G'(x) = f(x)$ とする.

G : f の原始関数
不定積分.

$$\left(F(x) - G(x)\right)' = f(x) - f(x) \equiv 0$$

$\rightsquigarrow$

$$F(x) - G(x) = C$$

$$F(x) = G(x) + C$$

$$\boxed{F(a) = 0}$$

$$= \int_a^a f(t)\,dt$$

レポート

$f' \equiv 0$

$\Rightarrow$ $f \equiv c$
定数
(証明)

$$\underset{\substack{\parallel \\ 0}}{F(a)} = G(a) + C \longrightarrow C = -G(a)$$

$$\underset{\substack{\parallel \\ \int_a^b f(t)\,dt}}{F(b)} = G(b) + C = G(b) - G(a)$$

$$\boxed{\int_a^b f(t)\,dt = G(b) - G(a)}$$

(1) $\int_0^1 e^x \, dx = ?$

(2) $\int_0^1 e^{2x} \, dx = ?$ $\qquad (e^{2x})' = 2e^{2x}$

(3) $\int_0^1 x^4 \, dx = ?$ $\qquad (x^5)' = \frac{1}{5}5x^4$