$X_1, X_2, \cdots$ 独立.

$\Leftrightarrow$ $X_1, \cdots, X_n$ が独立. (全ての $n$)

$\Leftrightarrow$ $X_{n_1}, X_{n_2}, \cdots, X_{n_\ell}$ : 独立

$(n_1 < n_2 < \cdots < n_\ell)$

---

注 $\int_\Omega |X| \, dP = +\infty$ の場合.

$E[X]$ は定義できない

$(\Omega, \mathcal{F}, P)$ : 確率空間.

---

独立の仮定なしで

$$E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$$

$$\rightsquigarrow E[X_1 + \cdots + X_n] = n\mu$$

$$E[S_n - n\mu] = 0 \rightsquigarrow E\left[\frac{S_n - n\mu}{\sigma\sqrt{n}}\right] = 0$$

$$V[X_n - \mu] = V[X_n]$$

独立

$$V[(X_1 - \mu) + \cdots + (X_n - \mu)]$$
$$= V[X_1 - \mu] + \cdots + V[X_n - \mu]$$
$$= n\sigma^2.$$

$$V\left[\frac{S_n - \mu}{\sigma\sqrt{n}}\right] = \frac{1}{\sigma^2 n} \cdot V[S_n - \mu] = \frac{n\sigma^2}{\sigma^2 n} =$$

$F_n(z) := P(Z_n \leq z)$

$Z_n$の分布関数

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{u^2}{2}} du$$

任意のzで連続.

$$= P(Z \leq z)$$

$Z \sim N(0,1)$

---

定理 任意の$z \in \mathbb{R}$

$$F_n(z) \longrightarrow F(z).$$

($Z_n \longrightarrow Z$ 分布収束)

---

$Z$: 確率変数.

$F(z) = P(Z \leq z)$

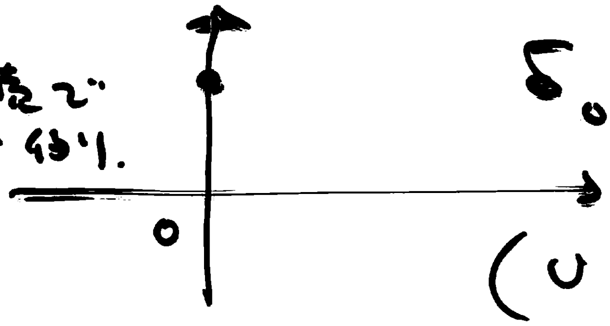
① $\lim_{\varepsilon \to +0} F(z+\varepsilon) = F(z)$ 右連続

①を示す 大事なポイント.

$$A_1 \supset A_2 \supset \cdots \quad \bigcap_{n=1}^{+\infty} A_n = A.$$

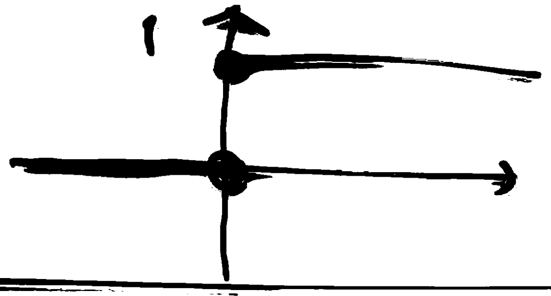
$$\Rightarrow \lim_{n \to \infty} P(A_n) = P(A)$$

た 連続でない例.



$$\delta_0(U) = \begin{cases} 0 & U \not\ni 0 \\ 1 & U \ni 0 \end{cases}$$

$(U \in \mathcal{B}_{\mathbb{R}})$



$\delta_0$ の $F(z)$

---

② $\lim_{z \to +\infty} F(z) = 1$, $\lim_{z \to -\infty} F(z) = 0$

任意に しぼ゜る.

---

$z_n \leq z$, $z > z + \varepsilon$

$-z < -z - \varepsilon$

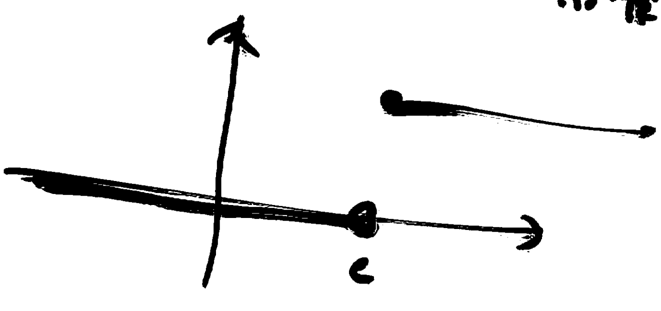
$z_n - z < z + (-z) - \varepsilon = -\varepsilon$

$z - z_n > \varepsilon$

---

$$\begin{array}{l} \Omega \longrightarrow \mathbb{R} \\ \omega \longmapsto c \end{array}$$

定数 $c$ の確率変数



分布関数.

$$E[e^{iz\xi}] = \int_\Omega e^{iz\xi(\omega)}\, dP(\omega)$$

$\omega \in \Omega$

$$\int_\Omega |e^{iz\xi}|\, dP = \int_\Omega dP(\omega) = 1 < +\infty$$

$\longrightarrow \int_\Omega e^{iz\xi} dP(\omega)$ が定義できる.

※. ずしも同一分布で無い

$E[X_i] = \mu$, $V[X_i] = \sigma^2$ 有限.

ここに使わない

前回の仮定

---

$X, Y$ : 独立. $Z = X + Y$.

$$\varphi_Z(\xi) = \varphi_X(\xi)\,\varphi_Y(\xi)$$

---

$Z_1 + \cdots + Z_n = U_n$

$$\varphi_{U_n}(\xi) = \varphi_{Z_1}(\xi) \cdots \varphi_{Z_n}(\xi)$$

$$\varphi_{aX}(\xi) = \varphi_X(a\xi)$$

$$F(a) = \int_\Omega f(a,\omega)\, dP(\omega)$$

$a \in \mathbb{R}$

$$\int_\Omega \left| \frac{\partial}{\partial a} f(a,\omega) \right| dP(\omega) < +\infty$$

$$\to \quad F'(a) = \int_\Omega \frac{\partial f}{\partial a}(a,\omega)\, dP(\omega)$$

---

仮定 $\displaystyle\int_\Omega |Z|\, dP(\omega) < +\infty$

$$\int_\Omega \left| \frac{\partial}{\partial \xi} e^{iZ\xi} \right| dP(\omega)$$

$$= \int_\Omega \left| iZ(\omega)\, e^{iZ(\omega)\xi} \right| dP(\omega)$$

$$= \int_\Omega |Z(\omega)|\, dP(\omega) < +\infty$$

$\to$ $\varphi_Z(\xi)$ は $\xi$ について微分可能.

$$\varphi'_Z(\xi) = \int_\Omega iZ(\omega)\, e^{iZ\xi}\, dP(\omega)$$

$$\varphi'_Z(0) = iE[Z] =$$

$$= E[e^{iZ0}]$$

---

$\varphi'(0) = i\mu, \quad \varphi(0) = E[1] = 1$

$$\frac{\varphi(\xi)-\overbrace{\varphi(0)}^{=1}}{\xi} \longrightarrow \varphi'(0)=\mu$$

$$\frac{\varphi(\xi)-\varphi(0)-\varphi'(0)\xi}{\xi} \longrightarrow 0 \quad (\xi \to 0)$$

$$\Updownarrow$$

$$\varphi(\xi) = \varphi(0) + \varphi'(0)\xi + \underbrace{o(\xi)}_{A}$$

$$\frac{A(\xi)}{\xi} \longrightarrow 0 \quad (\xi \to 0)$$

---

$\alpha \in \mathbb{R}$

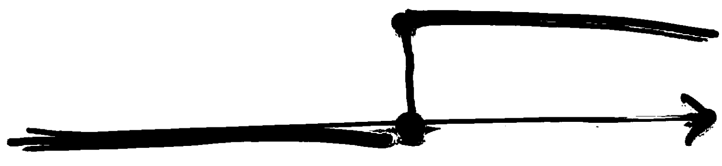
$$\left(1+\frac{\alpha}{n}\right)^n \longrightarrow e^{\alpha}$$

---

$\alpha \in \mathbb{C}$ でもOK.

$U \in \mathcal{B}_{\mathbb{R}}$

$$\delta_\mu(U) = \begin{cases} 1 & (\mu \in U) \\ 0 & (\mu \notin U) \end{cases}$$



$\mu$

$$\cancel{E[\mu]}\ E[e^{i\mu\xi}] = e^{i\mu\xi}\overbrace{E[1]}^{=1} = e^{i\mu\xi}$$