The Method of the Least Square

Prof. Dr. Nobuyuki TOSE

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  $\mathrm{Im}(A)$ is the subspace spaned (generaed) by $\vec{a}_1, \cdots, \vec{a}_n$

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- **Theorem** $(A\vec{x}, \vec{y}) = (\vec{x}, {}^tA\vec{y})$

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- **Problem** Find the minimum value of $||\vec{e} - A\vec{v}|| = ||\vec{e} - s\vec{a}_1 - t\vec{a}_2||$

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  when $A\vec{v}$ is the orthogonal projection of $\vec{e}$ on V.

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$$\Leftrightarrow \left({}^{t}A\left(\vec{e} - A\vec{v}\right), \vec{\beta}\right) = 0 \quad (\vec{\beta} \in \mathbb{R}^2)$$
$$\Leftrightarrow {}^{t}A\left(\vec{e} - A\vec{v}\right) = \vec{0}$$
$$\Leftrightarrow {}^{t}AA\vec{v} = {}^{t}A\vec{e}$$

Method of the least square

- $\vec{w} \in \mathrm{Im}(A) \quad \rightarrow \vec{w} = A\vec{v}$
- $(\vec{e} - A\vec{v}) \perp \mathrm{Im}(A)$

$$\Leftrightarrow \left(\vec{e} - A\vec{v}, A\vec{\beta}\right) = 0 \quad (\vec{\beta} \in \mathbb{R}^2)$$

$$\Leftrightarrow \left({}^t A\left(\vec{e} - A\vec{v}\right), \vec{\beta}\right) = 0 \quad (\vec{\beta} \in \mathbb{R}^2)$$

$$\Leftrightarrow {}^t A\left(\vec{e} - A\vec{v}\right) = \vec{0}$$

$$\Leftrightarrow {}^t AA\vec{v} = {}^t A\vec{e}$$

Method of the Least Square No2

- Orth. Proj. $\vec{w} = A\vec{v}$

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- ${}^tAA\vec{v} = {}^tA\vec{e}$

$$ {}^tAA = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} $$

Method of the Least Square No2

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- ${}^{t}A\vec{e} = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

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- $\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

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- ${}^tA\vec{e} = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$
- $\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \to \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Method of the Least Square No2

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$$ {}^tAA = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} $$

- ${}^tA\vec{e} = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$
- $\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$$ = \tfrac{1}{9} \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \tfrac{1}{9} \begin{pmatrix} -2 \\ 4 \end{pmatrix} $$

Method of the Least Square No2

- Orth. Proj. $\vec{w} = A\vec{v}$
- ${}^tAA\vec{v} = {}^tA\vec{e}$

$${}^tAA = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$$

- $${}^tA\vec{e} = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$
- $$\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \rightarrow \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$