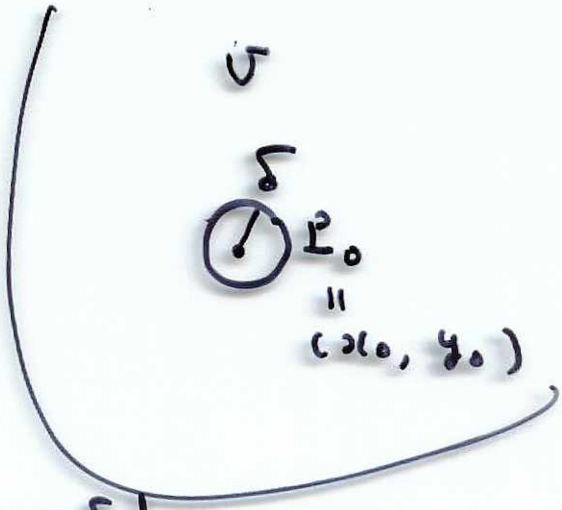
$f : U \longrightarrow \mathbb{R}$

$\uparrow$

$\mathbb{R}^2$の開集合



$B_\delta(P_0) \subset U$

$\shortparallel$

$\{ P \in \mathbb{R}^2 ;\ d(P, P_0) < \delta \}$

PとP_0の距離.

$= \{ (x, y) ;\ (x - x_0)^2 + (y - y_0)^2 < \delta^2 \}$

$P_0 \in U$で f が連続

$\overset{\Longleftrightarrow}{\text{def.}}$ $P_n \in U,\ P_n \to P_0\ (n \to +\infty) \Rightarrow f(P_n) \to f(P_0)$

$f(P_0) > 0$ とする.　$f : U \to \mathbb{R}$ 連続とする.

「連続関数は符号を急に変えない」

ある $\delta > 0$ が存在し

$Q \in B_\delta(P_0)$

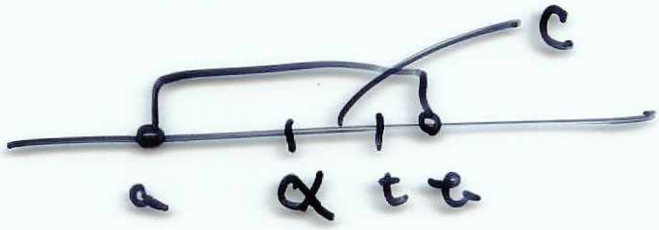
$\Rightarrow f(Q) > 0$

定理 8.2

1変数の場合の定理2.4.

Taylorの定理

$f : (a, b) \longrightarrow \mathbb{R}$　C^∞級.



$f(t) - f(\alpha)$

$$= f'(\alpha)(t-\alpha) + \frac{1}{2!} f''(c)(t-\alpha)^2$$

cはtとαの間のある数.　定理4.11.

定理

(1)　$f : U \longrightarrow \mathbb{R}$　$\boxed{C^2\text{級}}$ $\longrightarrow$ $f_{xy} = f_{yx}$

各点で偏微分可能.

f_x, f_y : 各点で ———

$f, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$
全て連続.

(2)　$f_x(a, b) = f_y(a, b) = 0$

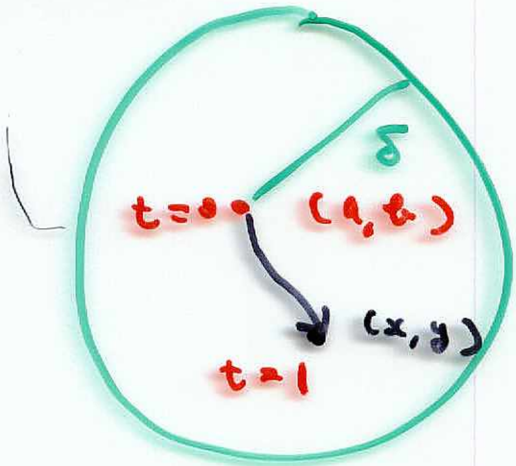
(3)　$f_{xx}(a, b) > 0$ (<0),

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0 \text{ at } (a, b)$$

$\Longrightarrow$ (a, b)で極小.
(大)

f_{xx} [redacted], $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ 連続.

ある $\delta > 0$ が存在して



$Q \in B_\delta(a, b)$

$\Rightarrow f_{xx}(Q) > 0$

$\quad \left|\quad\right|(Q) > 0$

$(x, y) \in B_\delta(a, b)$ $\qquad \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} x-a \\ y-b \end{pmatrix}$

$F(t) = f(a + \xi t, b + \eta t)$

$F(0) = f(a, b)$, $F(1) = f(x, y)$

$F'(t) = f_x(a + \xi t, b + \eta t) \cdot \xi$

$\qquad + f_y(\text{——}) \cdot \eta$

(a, b) は停留点

$F'(0) = \overset{=0}{\boxed{f_x(a, b)}} \cdot \xi + \overset{=0}{\boxed{f_y(a, b)}} \cdot \eta$

$= 0$

$F''(t) = f_{xx} \cdot \xi^2 + f_{xy} \cdot \xi\eta$

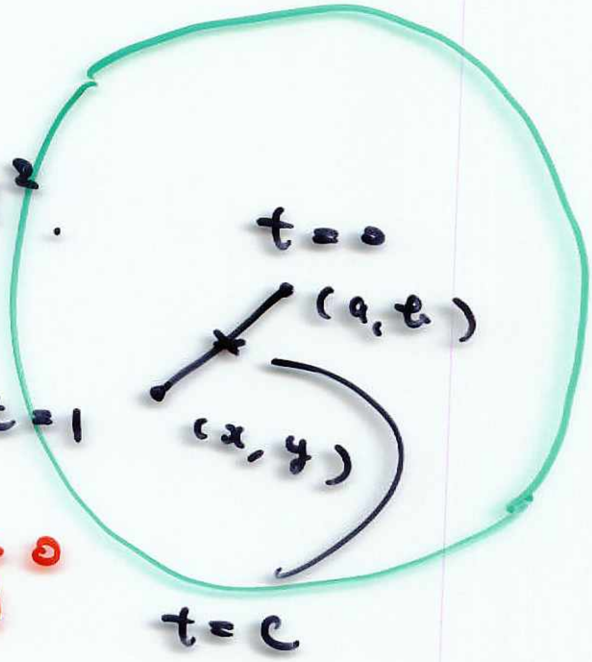
$\qquad + f_{yx} \cdot \eta\xi + f_{yy}\, \eta^2$

$= f_{xx}(\quad) \cdot \xi^2$

$\qquad + 2 f_{xy}\, \xi\eta + f_{yy} \cdot \eta^2$

$$F(1) - F(0) = \overbrace{F'(0)}^{=0} \cdot 1 + \frac{1}{2!} F''(c)\, 1^2 .$$

c は 0 と 1 の間



$$= \left(H(Q_c) \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right) > 0$$

$$H(Q) = \begin{pmatrix} f_{xx}(Q) & f_{xy}(Q) \\ f_{yx}(Q) & f_{yy}(Q) \end{pmatrix}$$

$$Q_t = (a + \xi t,\ b + \eta t)$$

$$f_{xx}(Q_c) > 0,\quad \det(H(Q_c)) > 0$$

$$\longrightarrow \quad f(x, y) - f(a, b) > 0$$

$$f(x, y) > f(a, b)$$

$$\left(\begin{array}{c} (x, y) \in B_\delta(a, b) \\ (x, y) \neq (a, b) \end{array} \right)$$

$\longrightarrow$ $f(a, b)$ は極小値.

定理 $\det(H(a,b)) < 0$ $\Longleftrightarrow$ $\alpha\beta < 0$

α, β は $H(a,b)$ の固有値.

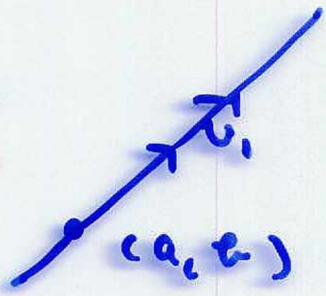
$f_x(a,b) = f_y(a,b) = 0$

$\Longrightarrow$ (a,b) は極小でも極大でもない

$H = H(a,b)$ (a,b) における Hesse 行列.

$$\left.\begin{aligned} H\vec{v}_1 &= \alpha \vec{v}_1, \quad \vec{v}_i \neq \vec{0} \\ H\vec{v}_2 &= \beta \vec{v}_2 \end{aligned}\right\} \text{固有ベクトルをとる}$$

$\alpha > 0$, $\beta < 0$ の場合を考える.



$F(t) = f(a + \xi t, b + \eta t)$

$F'(t) = \xi f_x(\quad)$ $\quad \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \vec{v}_1$

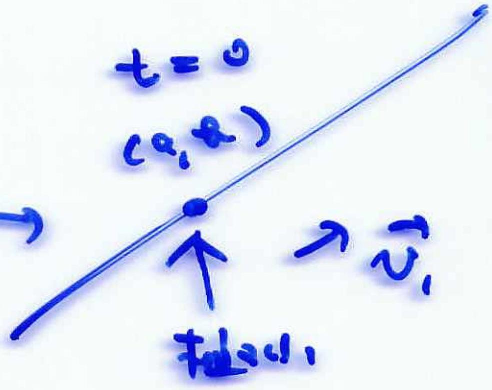
$\qquad + \eta f_y(\quad)$

$F'(0) = 0$

$F''(t) = \xi^2 f_{xx} + 2\xi\eta f_{xy} + \eta^2 f_{yy}$

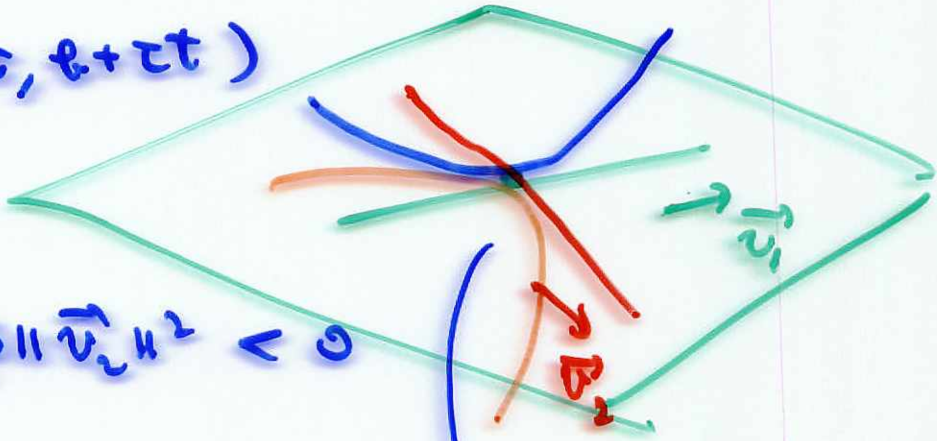
$$\begin{aligned} F''(0) &= \xi^2 f_{xx}(a,b) + 2\xi\eta f_{xy}(a,b) \\ &\quad + \eta^2 f_{yy}(a,b) \\ &= \left(H\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right) \\ &= (H\vec{v}_1, \vec{v}_1) = (\alpha \vec{v}_1, \vec{v}_1) \\ &= \alpha \underbrace{\|\vec{v}_1\|^2}_{>0} > 0 \end{aligned}$$

$$\left.\begin{array}{l} F'(0)=0 \\ F''(0)>0 \end{array}\right\} \longrightarrow$$



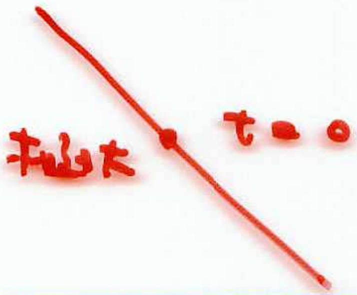
$$\vec{v}_2 = \begin{pmatrix} \sigma \\ \tau \end{pmatrix}$$

$$G(t) = f(a+\sigma t,\ b+\tau t)$$



$$G'(0) = 0$$

$$G''(0) = \cdots = \beta \|\vec{v}_2\|^2 < 0$$



(a, b) では
極大でも
極小でもない

$$z = x^3 + y^3 + 6xy$$

$$\begin{cases} z_x = 3x^2 + 6y = 3(x^2 + 2y) \\ z_y = 3y^2 + 6x = 3(y^2 + 2x) \end{cases}$$

$$z_x = z_y = 0 \iff (x, y) = (0, 0), (-2, -2)$$

$$z_{xx} = 6x, \quad z_{xy} = 6, \quad z_{yy} = 6y$$

$$H = \begin{vmatrix} 6x & 6 \\ 6 & 6y \end{vmatrix} = 36 \begin{vmatrix} x & 1 \\ 1 & y \end{vmatrix}$$
$$= 36(xy - 1)$$

---

at $(0, 0)$ $\quad H = 36 \cdot (0 - 1) = -36 < 0$

$\longrightarrow$ 極大でも・極小でもない

at $(-2, -2)$

$$H = 36 \cdot ((-2)(-2) - 1)$$
$$= 108 > 0$$
$$z_{xx}(-2, -2) = 6(-2) = -12 < 0$$

$\longrightarrow$ 極大.

$$z = (x^2 + y^2)^2 - 2(x^2 - y^2)$$

の極値を求めよ.