$f : U \longrightarrow \mathbb{R}$ 　　$U \subset \mathbb{R}^2$ 開集合.

$(a, b) \in U$ が停留点とする.

i.e. $f_x(a, b) = f_y(a, b) = 0$

$\Leftarrow$ ((a, b) でfが極値を持つ)

目標.

$f_x(a, b) = f_y(a, b) = 0$

$f_{xx}(a, b) > 0$ 　$<$ 　$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$

at (a, b)

$\Rightarrow$ (a, b) で極小.
大.

A 対称行列

$$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \quad {}^tA = A.$$

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = ax^2 + 2bxy + cy^2$$

2次形式.

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① $\Phi_A(\lambda) = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix}$

$= (\lambda - a)(\lambda - b) - c^2$

$= \lambda^2 - (a+b)\lambda + ab - c^2$

$D = (a+b)^2 - (4ab - c^2)$

判別式

$= (a-b)^2 + c^2 \geqq 0$

$$\boxed{\text{対称行列 } A \text{ は 2 実根を持つ.}}$$

Ich bin müde.

$D = 0 \iff a = b,\ c = 0$

$\iff A = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

② $D > 0$ の場合を考える. $\alpha \neq \beta$.

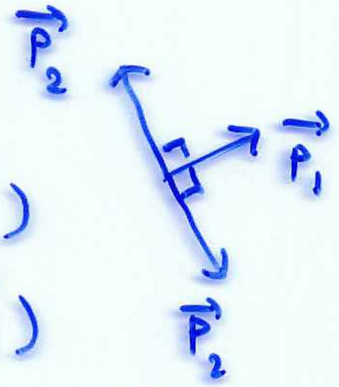
$\lambda = \alpha$ の固有ベクトル $A\overrightarrow{p_1} = \alpha\overrightarrow{p_1}$

$\|\overrightarrow{p_1}\| = 1$

$\lambda = \beta$ の固有ベクトル $A\overrightarrow{p_2} = \beta\overrightarrow{p_2}$

$\|\overrightarrow{p_2}\| = 1$

$$\boxed{(\vec{p}_1, \vec{p}_2) = 0}$$



$$(A\vec{p}_1, \vec{p}_2) = (\vec{p}_1, {}^tA\vec{p}_2) = (\vec{p}_1, A\vec{p}_2)$$

$$(A\vec{p}_1, \vec{p}_2) = \alpha(\vec{p}_1, \vec{p}_2)$$

$$(\vec{p}_1, A\vec{p}_2) = \beta(\vec{p}_1, \vec{p}_2)$$

$$\boxed{{}^tA = A}$$

$$\rightsquigarrow \quad \underset{\neq 0}{(\alpha - \beta)}(\vec{p}_1, \vec{p}_2) = 0$$

$$\rightsquigarrow \quad (\vec{p}_1, \vec{p}_2) = 0$$

$-\vec{p}_2$ をあらためて $\vec{p}_2$ とする.

$$P = (\vec{p}_1 \ \vec{p}_2) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

θ 回転

$$AP = P\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

P : 回転

${}^tP$ も回転

$$\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \left({}^tP A\begin{pmatrix} x \\ y \end{pmatrix}, {}^tP\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$${}^tP = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = P^{-1}$$

$$= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$(-\theta)$ 回転

$$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = \left({}^tPA\underbrace{\boxed{P\,{}^tP}}_{=I_2}\begin{pmatrix}x\\y\end{pmatrix}, {}^tP\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$${}^tP = P^{-1}$$

$${}^tPP = P\,{}^tP = I_2$$

$${}^tPAP = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$$

$${}^tP\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

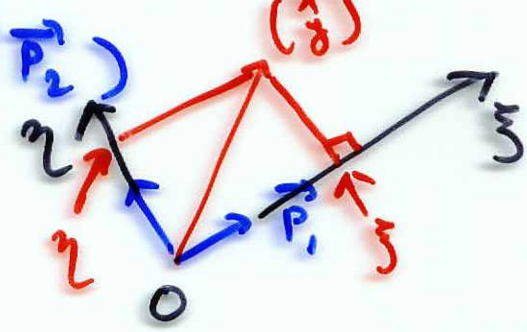
$$= \left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \begin{pmatrix}\xi\\ \eta\end{pmatrix}\right)$$

$$= \alpha\xi^2 + \beta\eta^2.$$

$${}^tP\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\xi\\ \eta\end{pmatrix} \qquad P = (\vec{p}_1\ \vec{p}_2)$$

$$P\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$$

$$= \xi\vec{p}_1 + \eta\vec{p}_2$$



$$\boxed{\begin{aligned} ax^2 + 2cxy &= \alpha\xi^2 + \beta\eta^2 \\ + by^2 & \end{aligned}}$$

$$\left(\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) > 0 \quad \begin{pmatrix}x\\y\end{pmatrix} \neq \vec{0}\right)$$

A : 対称行列.

$$\Longleftrightarrow \quad \alpha, \beta > 0$$

$$A = \begin{pmatrix}a & c\\ c & b\end{pmatrix}$$

$\Longleftrightarrow \quad a > 0$ かつ $\underbrace{ab - c^2}_{\text{\rotatebox{90}{=}}\ \det(A)} > 0$

定理 CT 241 p. $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ 対称.

(i) $\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) > 0$ $\quad\left(\begin{pmatrix} x \\ y \end{pmatrix} \neq \vec{0}\right)$
< 0

$\Longleftrightarrow$ (ii) A の固有値 $\alpha, \beta > 0$ $\quad \alpha, \beta < 0$

$\Longleftrightarrow$ (iii) $a > 0$, $ab - c^2 > 0$
< 0

(ii) $\Rightarrow$ $\left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \overset{0}{\overset{\wedge}{\alpha}}\xi^2 + \overset{0}{\overset{\wedge}{\beta}}\eta^2$.

$^tP\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$

$\xi \neq 0 \Rightarrow \alpha\xi^2 > 0$

$\alpha\xi^2 + \underbrace{\beta\eta^2}_{\geqq 0} > 0$

$\eta \neq 0 \Rightarrow \beta\eta^2 > 0$

$\underbrace{\alpha\xi^2}_{\geqq 0} + \beta\eta^2 > 0$

P : 正則.

$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \neq \vec{0} \Rightarrow \left(A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) > 0$

$\Updownarrow$ $\begin{pmatrix} x \\ y \end{pmatrix} \neq \vec{0}$ $\quad \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} \xi \\ \eta \end{pmatrix}}$

(i) $\Rightarrow$ (ii) $\alpha \leqq 0$ とする.

$(A\vec{p_1}, \vec{p_1}) = (\alpha\vec{p_1}, \vec{p_1})$ $\quad (\vec{p_1}, \vec{p_1}) = \|\vec{p_1}\|^2$

$= \alpha\|\vec{p_1}\|^2 = \alpha$

$\|\vec{p_1}\| = 1$ $\rightsquigarrow$ $\alpha > 0$

同様に $\beta > 0$

$(A\vec{v}, \vec{v}) > 0$
$(\vec{v} \neq \vec{0})$

(ii) $\alpha, \beta > 0 \iff \alpha\beta > 0$ かつ

$\alpha + \beta > 0$

(iii) $a > 0, \quad ab - c^2 > 0$

$\lambda^2 - (a+b)\lambda + (ab - c^2) =$

$= (\lambda - \alpha)(\lambda - \beta)$

$\rightsquigarrow \quad \alpha + \beta = a + b, \quad \alpha\beta = ab - c^2$

解と係数の関係.

(ii) $\alpha\beta > 0 \rightsquigarrow ab - c^2 > 0 \qquad a, b > 0$

$\alpha + \beta > 0 \rightsquigarrow \boxed{a + b > 0}$

$ab \geqq ab - c^2 > 0 \rightsquigarrow \boxed{ab > 0}$

(i) $ab - c^2 > 0 \rightsquigarrow \alpha\beta > 0$

$ab - c^2 > 0 \rightsquigarrow ab > 0 \rightsquigarrow b > 0$

$\wedge$

$ab \qquad \boxed{a > 0}$

$a + b = \alpha + \beta > 0$

コメント　$A : m \times n$

$$\begin{array}{cc} & n \\ m & \begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{pmatrix} \end{array}$$

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n \in \mathbb{R}^m$$

$\vec{x} \in \mathbb{R}^n,\ \vec{y} \in \mathbb{R}^m$

$$\boxed{(A\vec{x}, \vec{y}) = (\vec{x}, {}^tA\vec{y})}$$

$$\begin{array}{cc} {}^tA = \begin{pmatrix} {}^t\vec{a}_1 \\ \vdots \\ \\ \end{pmatrix} & n \\ m & \end{array}$$

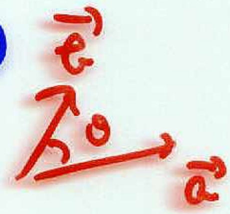
$n \times m$

$${}^tA = \begin{pmatrix} {}^t\vec{a}_1 \\ {}^t\vec{a}_2 \\ \vdots \\ {}^t\vec{a}_n \end{pmatrix}$$

→ 証明は宿題.

$(P\vec{v}, P\vec{w}) = (\vec{v}, \vec{w})$　$P : 2 \times 2$

（全ての $\vec{v}, \vec{w} \in \mathbb{R}^2$）



P: 回転ならばOK.

$(\vec{a}, \vec{b}) = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta$

$$(P\vec{v}, P\vec{w}) = (\vec{v}, \vec{w})$$

$$\parallel$$

$$({}^tPP\vec{v}, \vec{w}) = (\vec{v}, \vec{w})$$

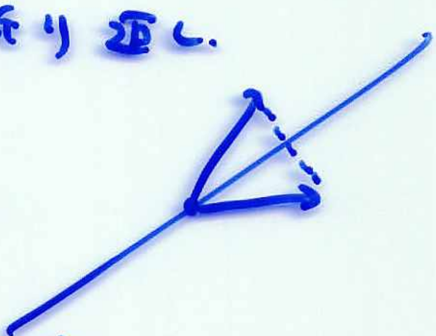
$$(\underbrace{{}^tPP\vec{v} - \vec{v}}_{=\vec{0}}, \vec{w}) = 0$$ 全ての $\vec{w}$ でOK.

$\rightsquigarrow ({}^tPP - I_2)\vec{v} = \vec{0}$ (全ての $\vec{v}$)

$\rightsquigarrow ({}^tPP - I_2) = O_2$

$\rightsquigarrow {}^tPP = I_2$ 以下平面

P 回転 OR 折り返し.



$A : 2\times 2$ 全ての $\begin{pmatrix} x \\ y \end{pmatrix}$ に対して

$A\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Longrightarrow A = O_2$

$A = (\vec{a}_1\ \vec{a}_2)$

$(\vec{a}_1\ \vec{a}_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{a}_1 = \vec{0}$

$(\vec{a}_1\ \vec{a}_2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{a}_2 = \vec{0}$

$\Downarrow$

$A = (\vec{0}\ \vec{0}) = O_2$

$A = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$ を 回転行列で
対角化する

$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$A = P \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} {}^tP$$

まで.