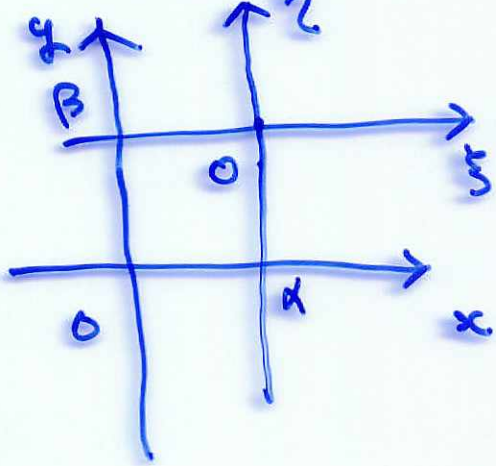
$$z = x^2 + xy + y^2 - 4x - 2y$$

の停留点を求める.

$$\begin{cases} z_x = 2x + y - 4 = 0 \\ z_y = x + 2y - 2 = 0 \end{cases}$$

$x = 2,\ y = 0.$



$\xi = x - \alpha,\ \eta = y - \beta$

α, β を適当に選んで

$$z = \xi^2 + \xi\eta + \eta^2 + \gamma$$

$(\gamma :$ 定数$)$

とできる.

$$(x-\alpha)^2 + (x-\alpha)(y-\beta) + (y-\beta)^2 + \gamma$$

$$= x^2 + xy + y^2 - 4x - 2y.$$

展開してできる（各自）

$$\begin{cases} 2(x-\alpha) + (y-\beta) = 2x + y - 4 \\ (x-\alpha) + 2(y-\beta) = x + 2y - 2 \end{cases}$$

$$\begin{cases} -2\alpha - \beta = -4 \\ -\alpha - 2\beta = -2 \end{cases} \rightarrow \begin{cases} 2\alpha + \beta = +4 \\ \alpha + 2\beta = 2 \end{cases}$$

$\alpha = 2,\ \beta = 0$

$$\gamma = z(x = 2, y = 0)$$
$$= 4 - 8 = -4.$$

$\xi = x - 2,\ \eta = y$ によって

$$z = \xi^2 + \xi\eta + \eta^2 - 4$$

対称行列 A の 2次形式.

$$= \left(\underbrace{A\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \begin{pmatrix}\xi\\ \eta\end{pmatrix}}_{=\begin{pmatrix}\xi+\frac{1}{2}\eta\\ \frac{1}{2}\xi+\eta\end{pmatrix}\cdot\begin{pmatrix}\xi\\ \eta\end{pmatrix}} \right) - 4$$

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

対称行列

$\xi \quad \eta$

$$\boxed{\text{対称行列は回転行列で対角化できる.}}$$

$$\Phi_A(\lambda) = \begin{vmatrix} \lambda-1 & -\frac{1}{2} \\ -\frac{1}{2} & \lambda-1 \end{vmatrix} = 0$$

固有値.

$$(\lambda-1)^2 - \left(\tfrac{1}{2}\right)^2 = 0 \qquad \lambda = 1 \pm \tfrac{1}{2}$$

$$\tfrac{1}{2},\ \tfrac{3}{2}.$$

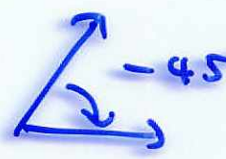
$$\lambda = \tfrac{1}{2} \quad \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = \vec{0} \iff s + t = 0$$

$$\lambda = \tfrac{3}{2} \quad \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = \vec{0} \iff s - t = 0$$

$$\lambda = \tfrac{1}{2} \quad s\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda = \tfrac{3}{2} \quad s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{p}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P = (\vec{p}_1, \vec{p}_2) = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$



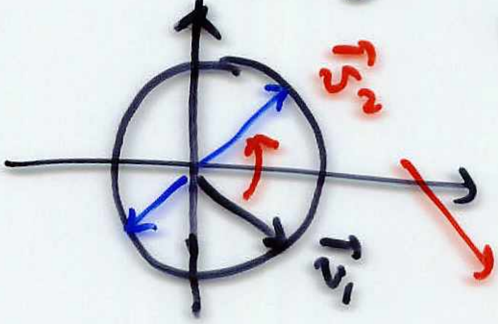
$-45°$ 回転

A：対称，$\alpha \neq \beta$ 固有値

$\vec{v}_1, \vec{v}_2$: それぞれの固有ベクトル

$\Longrightarrow \quad \vec{v}_1 \perp \vec{v}_2$

$\|\vec{v}_1\| = \|\vec{v}_2\| = 1$



$\longrightarrow (\vec{v}_1, \vec{v}_2)$ 回転

$$AP = (A\vec{p}_1 \ A\vec{p}_2) = \left(\tfrac{1}{2}\vec{p}_1 \ \tfrac{3}{2}\vec{p}_2\right)$$
$$= (\vec{p}_1 \ \vec{p}_2)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} = P\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

P^{-1}も回転　$(P^{-1}\vec{v}, P^{-1}\vec{w}) = (\vec{v}, \vec{w})$

$\longrightarrow$ 回転は内積を変えない。

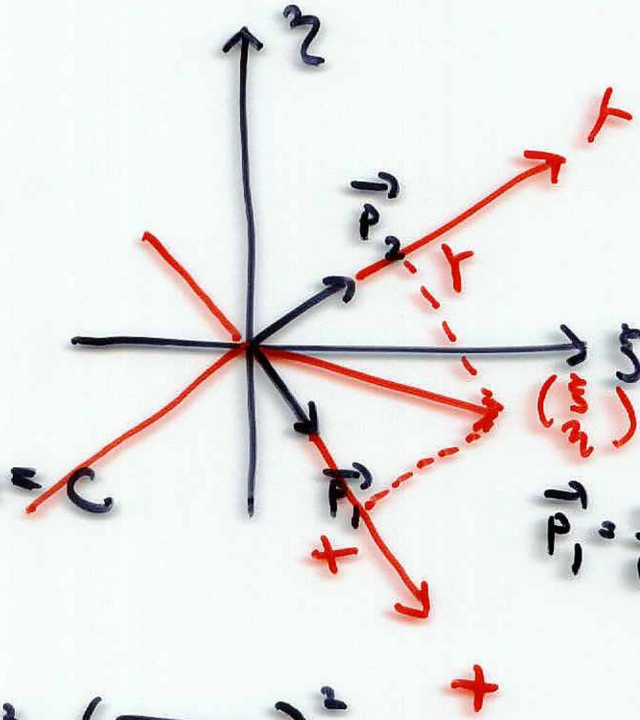
$$z = \left(A\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) - 4$$
$$= \Big(\underbrace{P^{-1}AP}_{= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}}\,P^{-1}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, P^{-1}\begin{pmatrix} \xi \\ \eta \end{pmatrix}\Big), \quad P P^{-1} = I_2$$

$$P^{-1}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$= \underbrace{\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix}\right)}_{-4} = \frac{1}{2}X^2 + \frac{3}{2}Y^2 - 4.$$

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = P\begin{pmatrix}X\\ Y\end{pmatrix} = (\vec{p}_1\ \vec{p}_2)\begin{pmatrix}X\\ Y\end{pmatrix}$$
$$= X\vec{p}_1 + Y\vec{p}_2$$

$$z = \frac{1}{2}X^2 + \frac{3}{2}Y^2 - 4.$$



$$c > 4$$

$$z = \frac{1}{2}X^2 + \frac{3}{2}Y^2 - 4 = c$$

$$\vec{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$\downarrow$$

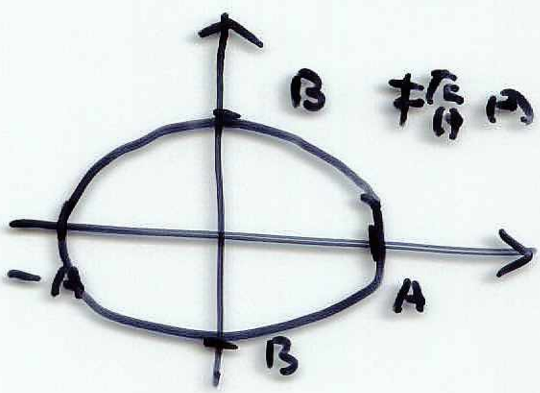
$$\frac{1}{2}X^2 + \frac{3}{2}Y^2 = \left(\sqrt{4+c}\right)^2$$

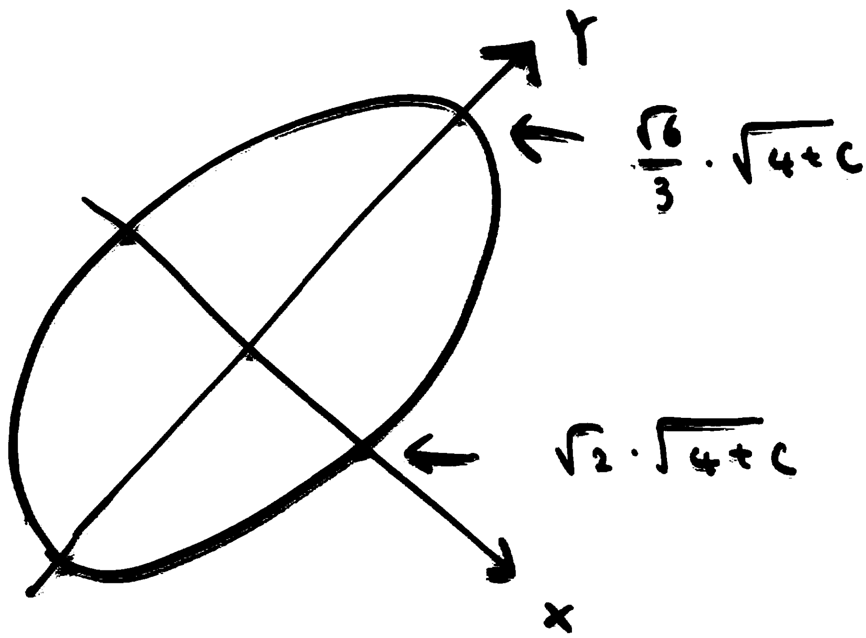
$$\sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$$

$$\left(\frac{X}{\sqrt{2}\cdot\sqrt{4+c}}\right)^2 + \left(\frac{Y}{\frac{\sqrt{6}}{3}\sqrt{4+c}}\right)^2 = 1$$

$A, B > 0$

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

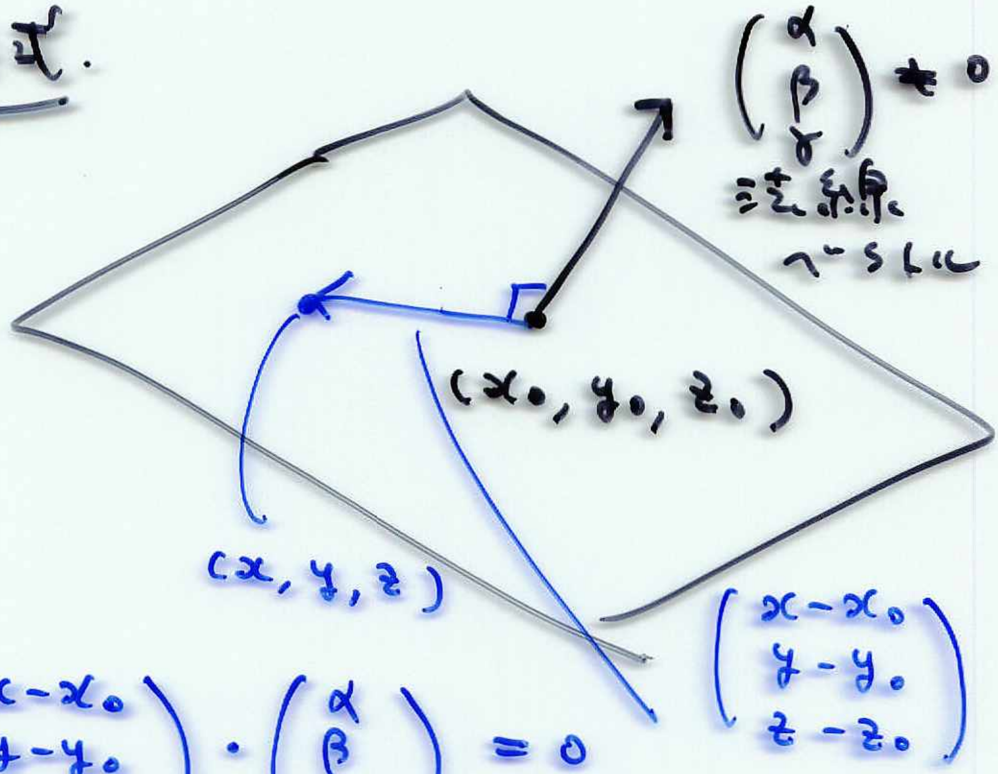




$\rightsquigarrow$ z は $\left(\begin{array}{l} X = Y = 0 \\ \xi = 0,\ \eta = 0 \\ x = 2,\ 0 \end{array} \right)$

において極小。

平面の方程式.



$$\begin{pmatrix}x-x_0\\y-y_0\\z-z_0\end{pmatrix}\cdot\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=0$$

$$\Leftrightarrow \quad \alpha(x-x_0)+\beta(y-y_0)+\gamma(z-z_0)=0$$

$3x-2y+z=1$ $\qquad x=0,\ y=0$

$-)\ 3\cdot 0-2\cdot 0+1=1$ $\qquad z=1$

$3x-2y+(z-1)=0$

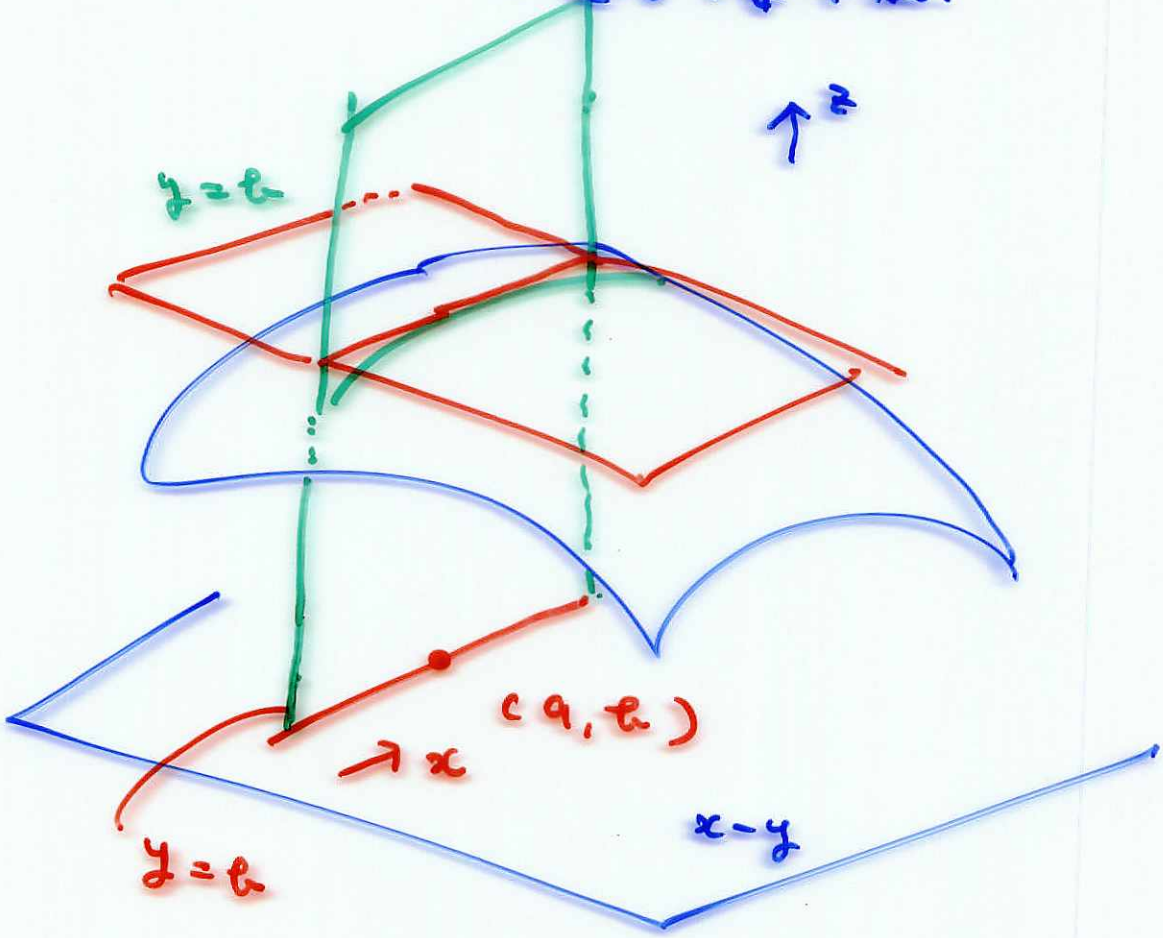
$$\Updownarrow$$

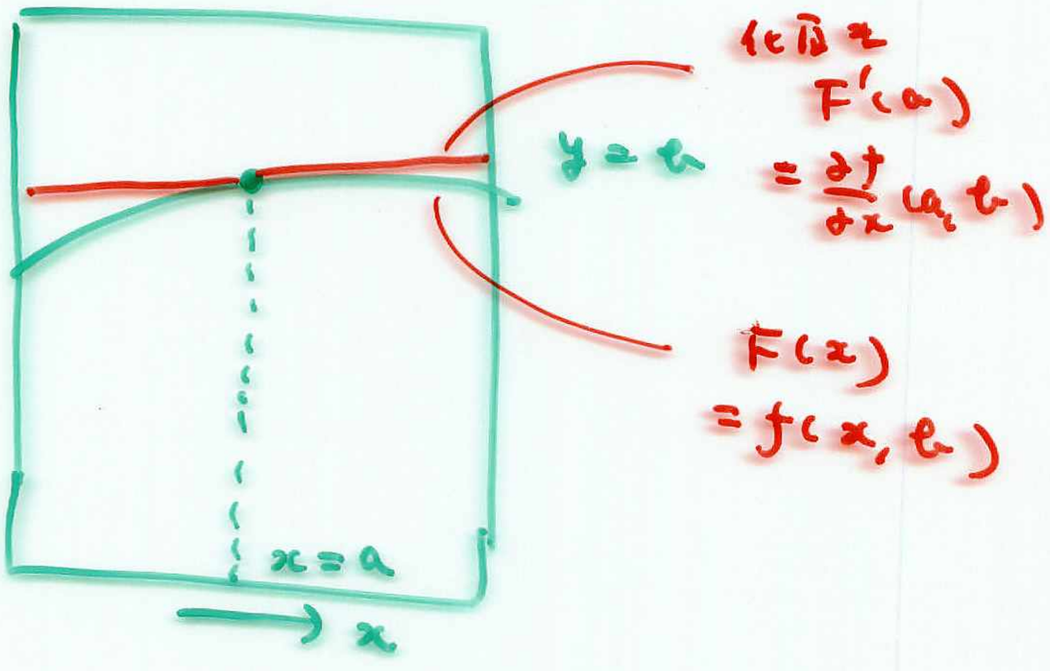
$$\begin{pmatrix}3\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z-1\end{pmatrix}=0$$

$(0,0,1)$ を通り 法線ベクトル $\begin{pmatrix}3\\-2\\1\end{pmatrix}$ の平面を表す.

$z = f(x, y)$

$(a, b, c = f(a, b))$
を通る接平面.





$$z = A(x-a) + B(y-b) + f(a,b)$$

$(a, b, f(a,b))$ を通る平面

$y = b$ の断面.

$$z = A(x-a) + f(a,b)$$

⇝ 傾き A

x-z 平面で

$\longrightarrow$ $A = \dfrac{\partial f}{\partial x}(a,b)$

同様に $B = \dfrac{\partial f}{\partial y}(a,b)$

'公式'

$$z = f_x(a,b)\cdot(x-a) + f_y(a,b)(y-b) + f(a,b)$$

接平面の方程式.

問 $z = x^3 + y^3 + 6xy$

の停留点を求める.

2週間後まで

例 8.11

$$z = x^2 + 4xy + 2y^2 - 6x - 8y.$$

平行移動と回転で簡単に.