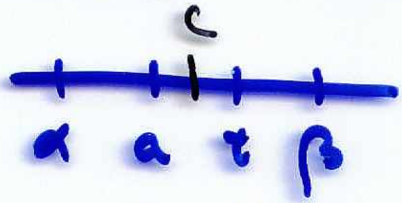
$$e^{ix} = \cos x + i \sin x.$$

Taylorの定理.　　CT 126 p.

$$f : (\alpha, \beta) \longrightarrow \mathbb{R} \quad C^{\infty}級.$$



$$f(t) = f(a) + f'(a)(t-a) + \frac{1}{2!} f''(a)(t-a)^2$$
$$+ \frac{1}{3!} f^{(3)}(a)(t-a)^3 + \cdots\cdots$$
$$\cdots + \frac{f^{(n)}(a)}{n!}(t-a)^n$$
$$+ \underbrace{f^{(n+1)}(c)\frac{(t-a)^{n+1}}{(n+1)!}}_{剰余項}$$

を満たす c が a と t の間にある.

注.

$f(t)=e^t$, $f'(t)=e^t$, $\cdots$, $f^{(n)}(t)=e^t$

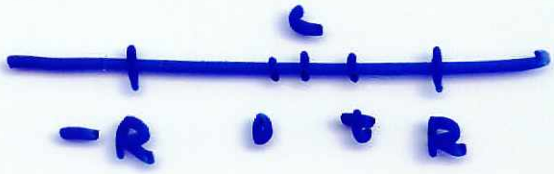
$$f^{(n)}(0)=1$$

$\underline{a=0}$

$$e^t = 1+t+\frac{1}{2!}t^2+\cdots+\frac{1}{n!}t^n+\frac{e^c}{(n+1)!}t^{n+1}$$

cは 0とtの間のある数.

$|t| \leq R$.



$$e^c \leq e^R$$

$$|剰余項| \leq \frac{e^R}{(n+1)!}R^{n+1}$$

- $\displaystyle\lim_{n\to+\infty}\frac{R^n}{n!}=0$ 　CT 129 page にあり. 脚注.

$n\to+\infty$

$$e^t = 1+t+\frac{1}{2!}t^2+\cdots\cdots$$

$$= \sum_{n=0}^{+\infty}\frac{1}{n!}t^n \quad (0!=1 とする)$$

CT 132 p. $a = 0$ とする.

$$f(t) = \sin t,\quad f' = \cos t,\quad f'' = -\sin t$$

$$f^{(3)} = -\cos t,\quad f^{(4)} = \sin t$$

$$f^{(4k)} = \sin t,\quad f^{(4k+1)} = \cos t$$

$$f^{(4k+2)} = -\sin t,\quad f^{(4k+3)} = -\cos t$$

$$f^{(4k)}(0) = 0,\quad f^{(4k+1)}(0) = 1,\quad f^{(4k+2)} = 0$$

$$f^{(4k+3)} = -1$$

$$\sin t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + - \cdots$$

$$+ \frac{f^{(n+1)}(c)}{n+1!}\, t^{n+1}$$

$|t| \le R$

$$|\text{剰余項}| \le \frac{R^{n+1}}{(n+1)!} \to 0 \quad (n \to +\infty)$$

$$\rightsquigarrow \sin t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + - \cdots$$

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + - \cdots$$

$t \in \mathbb{R}$

$e^{it} := \ ?$ $\qquad i = \sqrt{-1}$ 虚数単位.

$$e^{t} = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots$$

$$\begin{aligned}
e^{it} &:= 1 + it + \frac{1}{2!}(-t^2) + \frac{1}{3!}(-it^3) \\
&\quad + \frac{1}{4!}t^4 + \frac{1}{5!}it^5 + \frac{1}{6!}(-t^6) + \frac{1}{7!}(-it^7) \\
&\quad \underbrace{+ \cdots\cdots}_{\text{収束.}} \\
&= \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + - \cdots\right) \\
&\quad + i\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + - \cdots\right) \\
&= \cos t + i \sin t
\end{aligned}$$

$$\boxed{e^{it} = \cos t + i \sin t}$$

$$F(t) = f(t) + i\,g(t)$$

$$\underbrace{f(t),\ g(t)}_{\text{実数値.}} \in \mathbb{R}$$

$$F'(t) := f'(t) + i\,g'(t)$$

$$e^{(a+ib)t} = e^{at} \cdot e^{ibt} \qquad a, b \in \mathbb{R}$$

$$= e^{at}(\cos bt + i \sin bt)$$

確認.

$$(F(t)G(t))' = F'(t)G(t) + F(t)G'(t)$$

Leibniz rule は $\mathbb{C}$ 値でも ok.

$$(e^{(a+ib)t})' = (e^{at} \cdot e^{ibt})'$$

$$= (e^{at})' e^{ibt} + e^{at} \cdot (e^{ibt})'$$

$$(e^{ibt})' = (\cos bt + i \sin bt)'$$

$$= (\underbrace{-b}_{= i \cdot ib} \sin bt + i b \cos bt)$$

$$= ib(\cos bt + i \sin bt)$$

$$= ib \; e^{ibt}$$

$$= a\, e^{at} e^{ibt} + e^{at}\, ib\, e^{ibt}$$

$$= (a+ib)\, e^{at} e^{ibt}$$

<u>i.a.</u> $\alpha = a + ib \in \mathbb{C}$

$$(e^{\alpha t})' = \alpha e^{\alpha t}$$

定理 $F'(t) = \alpha F(t)$ $(\alpha \in \mathbb{C})$

$$\Rightarrow F(t) = e^{\alpha t} F(0)$$

$$(e^{-\alpha t} F(t))' = (e^{-\alpha t})' F(t) + e^{-\alpha t} F'(t)$$

$$= (-\alpha) e^{-\alpha t} F(t) + e^{-\alpha t} \alpha F(t)$$

$$\equiv 0$$

$\rightsquigarrow$ $e^{-\alpha t} F(t) \equiv C$ 定数.

$t=0$ $e^{0} F(0) = C \to F(0) = C$

$$\begin{aligned} &G'(t) \equiv 0 \rightsquigarrow \alpha'(t) \equiv 0,\ \beta'(t) \equiv 0 \\ &\ \shortparallel \\ &\alpha(t) + i\beta(t) \rightsquigarrow \alpha(t) = c_1 \\ &\qquad\qquad\qquad\quad\ \beta(t) = c_2 \\ &\qquad G(t) = c_1 + i c_2 \text{ 定数} \end{aligned}$$

$$F(t) = e^{\alpha t} C = F(0) e^{\alpha t}$$

$e^{\alpha t} \cdot e^{\beta t} = e^{(\alpha+\beta) t}$

特に $e^{\alpha t} \cdot e^{-\alpha t} = e^{0 t} = 1$

すぐには....

$$e^{ict}\, e^{idt} \qquad c, d \in \mathbb{R}$$

$$= (\cos ct + i \sin ct)(\cos dt + i \sin dt)$$

$$= (\cos ct \cos dt - \sin ct \sin dt)$$

$$+ i (\cos ct \sin dt + \cos dt \sin ct)$$

$$= \cos (c+d)t + i \sin (c+d)t$$

<u>公式</u> $\quad e^{ict} \cdot e^{idt} = e^{i(c+d)t}$

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$$

$$(\theta_1, \theta_2 \in \mathbb{R})$$

$$e^{\alpha t} \cdot e^{\beta t} = e^{at} e^{ibt} e^{ct} e^{idt}$$

$$= \underset{\uparrow}{\cdots} = e^{(\alpha+\beta)t}$$

各自. 計算.

$f'' + Af' + Bf = 0$ 微分方程式

特性方程式

$$\lambda^2 + A\lambda + B = 0$$

$D = A^2 - 4B > 0$ 　2解 $\alpha, \beta \in \mathbb{R}$
$\alpha \neq \beta$.

$D = 0$ 後述.

$D < 0$ (今日)

$$f'' + f' + f = 0$$

$$\lambda^2 + \lambda + 1 = 0 \qquad \lambda = \frac{-1 \pm \sqrt{3}\, i}{2}$$

$$f'' - (\alpha + \beta) f' + \alpha\beta f = 0 \qquad \alpha = \frac{-1 + \sqrt{3}\, i}{2}$$

$$\beta = \frac{-1 - \sqrt{3}\, i}{2}$$

$$\cancel{(f' - \alpha f)' = \beta (f' - \alpha f)}$$

$$\begin{cases} (f' - \alpha f)' = \beta (f' - \alpha f) \\ (f' - \beta f)' = \alpha (f' - \beta f) \end{cases}$$

$$f' - \alpha f = e^{\beta t} \left(f'(0) - \alpha f(0) \right)$$

$$-)\; f' - \beta f = e^{\alpha t} \left(f'(0) - \beta f(0) \right)$$

$$\underbrace{(\beta - \alpha)}_{-\sqrt{3}\, i} f = \cancel{e^{\beta t} f'(0)} -$$

$$(\beta - \alpha) f = e^{\beta t}(f'(0) - \alpha f(0)) - e^{\alpha t}(f'(0) - \beta f(0))$$

$$f = f'(0)\,\frac{e^{\beta t} - e^{\alpha t}}{\beta - \alpha} + f(0)\,\frac{\beta e^{\alpha t} - \alpha e^{\beta t}}{\beta - \alpha}$$

続きは web においておく。

CT342 page.