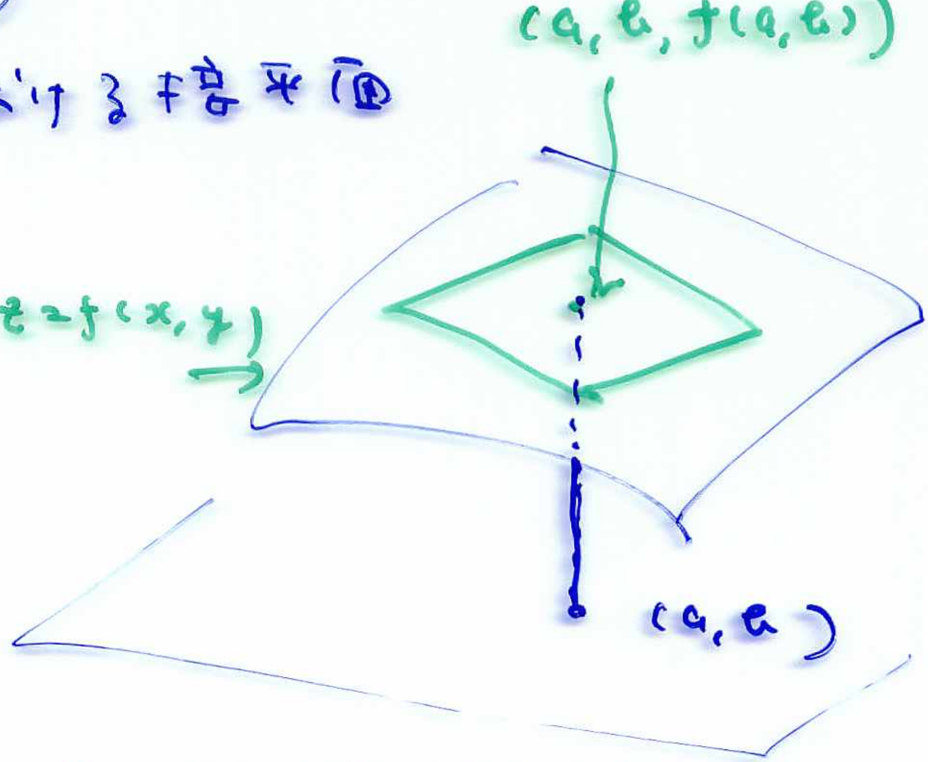
$z = f(x, y)$

(a, b) における接平面



$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

接平面

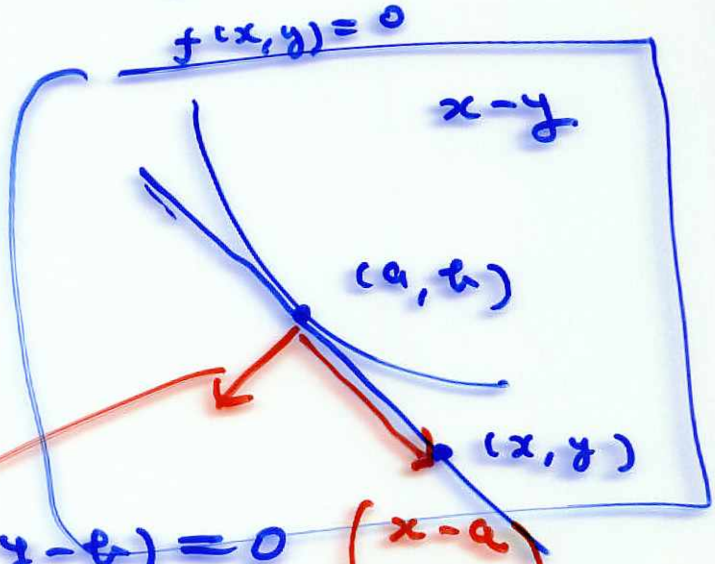
$z = f_x(a,b)(x-a) + f_y(a,b)(y-b)$

と

$z=0$ の交わり

$$\begin{pmatrix} f_x(a,b) \\ f_y(a,b) \end{pmatrix} \cdot \begin{pmatrix} x-a \\ y-b \end{pmatrix} = 0$$

$\rightsquigarrow$ $f_x(a,b)(x-a) + f_y(a,b)(y-b) = 0$



<u>例</u>

$f(x,y) = x^2 + y^2 - 1$

$f_x = 2x, \quad f_y = 2y$

$2a(x-a) + 2b(y-b) = 0$ $\quad \begin{pmatrix} x-a \\ y-b \end{pmatrix}$

$$\nabla(f)(a,b) = \begin{pmatrix} f_x(a,b) \\ f_y(a,b) \end{pmatrix}$$

法線方向の 1>

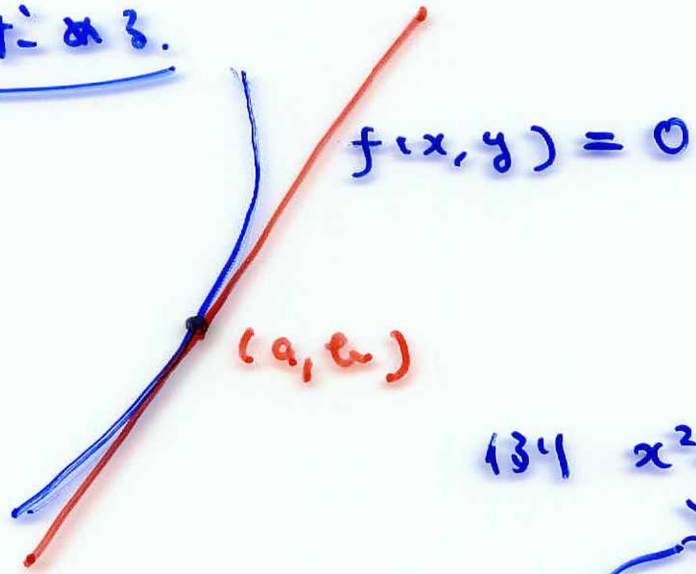
normal direction

勾 こう配ベクトル, gradient of f at (a,b)

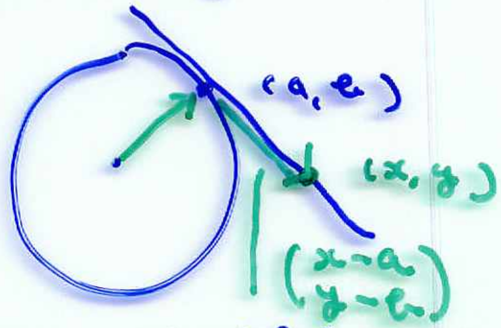
nabla

CT 273 page.

接線を求める.

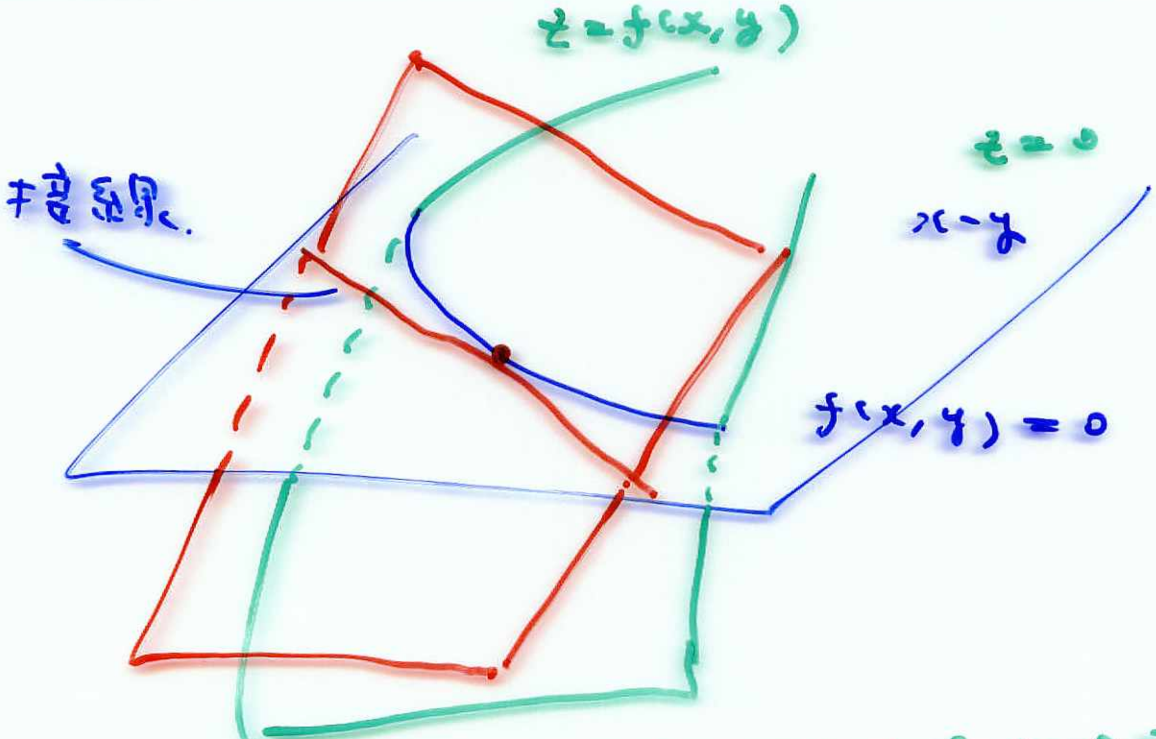


例 $x^2+y^2-1=0$



$$2a(x-a)+2b(y-b)=0$$

$$\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}x-a\\y-b\end{pmatrix}=0$$

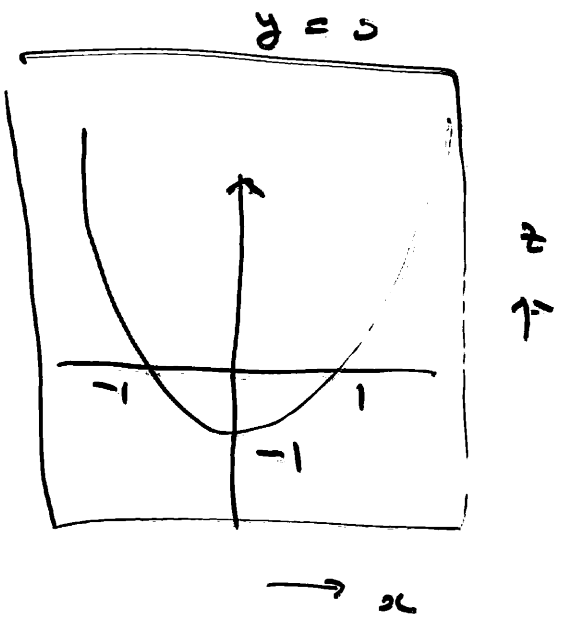


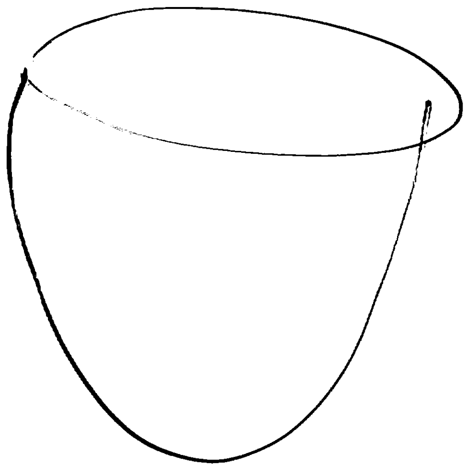
接平面 $z=f_x(a,b)(x-a)+f_y(a,b)(y-b)$

$+\ 0$

$f(a,b)=0$

$z = x^2 + y^2 - 1$





$$(x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \to z = 0$$

$$z_x = 2x, \quad z_y = 2y.$$

$$z = \sqrt{2}\left(x - \frac{\sqrt{2}}{2}\right) + \sqrt{2}\left(y - \frac{\sqrt{2}}{2}\right) + \underset{\substack{\uparrow \\ z = 0}}{0}$$

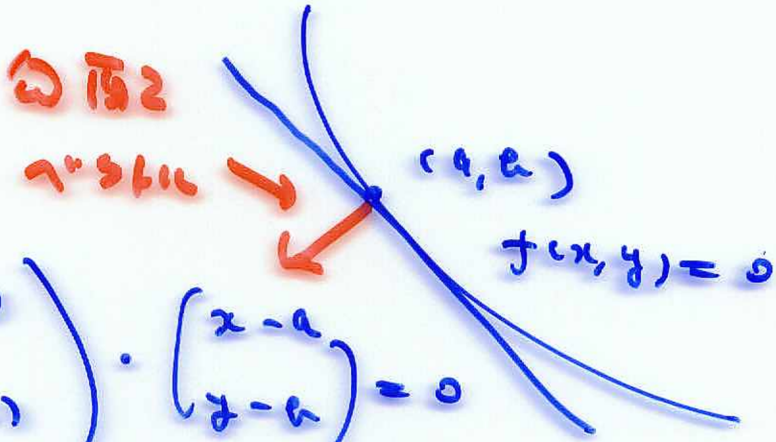
$z = x^2 + y^2 - 1$ の

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$ における接平面.

まとめ,

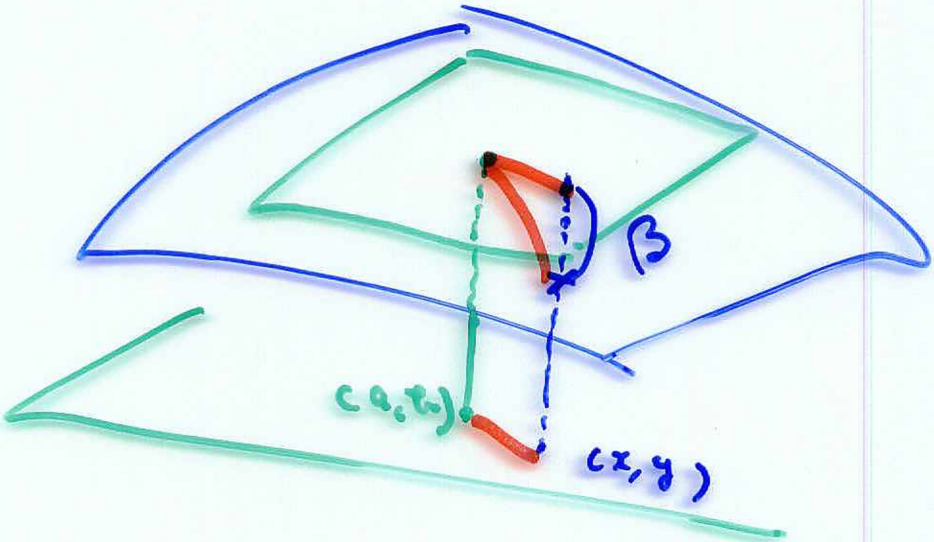
接線

$$\begin{pmatrix} f_x(a,b) \\ f_y(a,b) \end{pmatrix} \cdot \begin{pmatrix} x-a \\ y-b \end{pmatrix} = 0$$



275 page 全微分可能. 緑本 27 page 定理3.

$$\beta(x,y) := f(x,y) - \underbrace{\left(f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)\right)}_{\text{接平面の方程.}}$$



定義

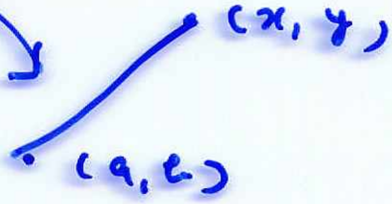
f が (a, b) で全微分可能

$$\Longrightarrow \quad \frac{\beta(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} \longrightarrow 0$$

$$\left((x,y) \to (a,b) \right)$$

$0 \longleftarrow$ $(x,y)\to(a,b)$

(β の方が $\sqrt{(x-a)^2+(y-b)^2}$ より速く 0 に行く)



定理

f が U 上定義されていて、

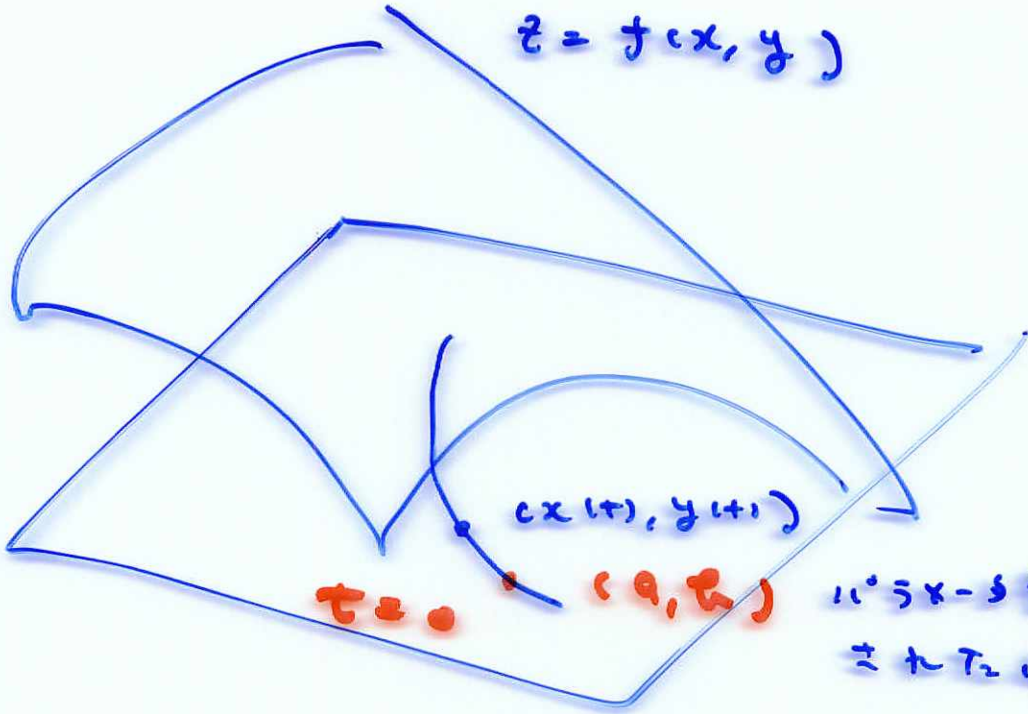
1) f : U の各点で x と y について偏微分できる。

2) f, f_x, f_y : 連続

$\Longrightarrow$ f は U の各点で全微分可能。

このとき f は U 上 C^1 級。

$$\beta(x,y) = f(x,y) - \left(f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b) \right)$$

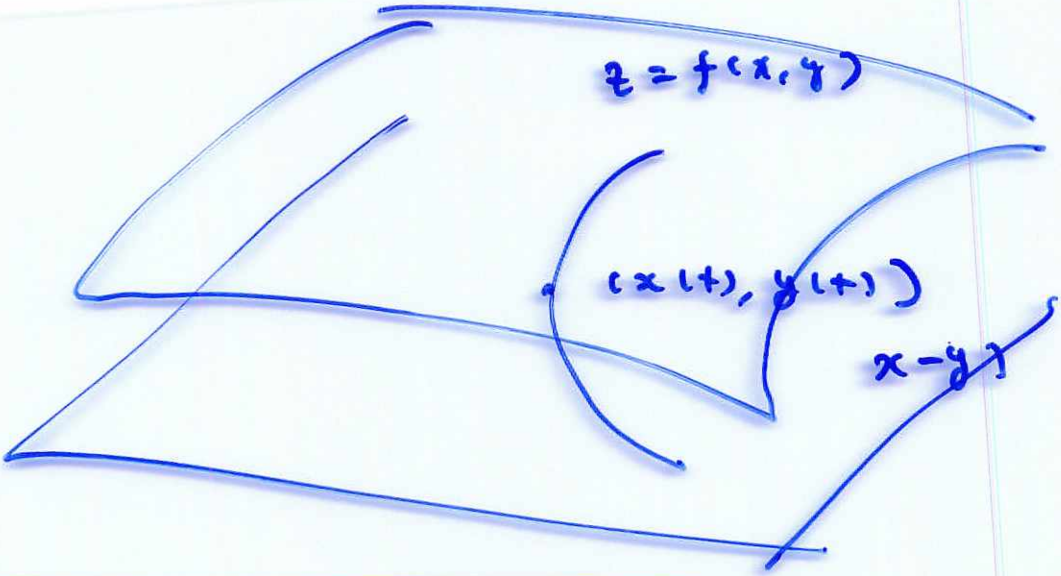


パラメータ表示された曲線

$$f(x, y) = f_x(a, b)(x-a) + f_y(a, b)(y-b) + f(a, b) + \beta(x, y)$$

$$\frac{\beta(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} \to 0$$

$$\left((x, y) \to (a, b) \right)$$



$F(t) := f(x(t), y(t))$ に対し〜

$f : C^1$ 級, 特に全微分可　$x(t), y(t)$：微分可能

$$F'(t) = \,?$$

$x(0) = a,\ y(0) = b.$　$F'(0) = \,?$　(T 276 p.

$$\frac{F(t)-F(0)}{t-0} = \frac{f(x(t), y(t)) - f(a,b)}{t}$$

$$= \frac{f_x(a,b)\,(x(t)-\overset{x(0)}{a})}{t} + \frac{f_y(a,b)\,(y(t)-\overset{y(0)}{b})}{t}$$

$$+ \frac{\beta(x(t), y(t))}{t}$$

$$= f_x(a,b)\overbrace{\boxed{\frac{x(t)-x(0)}{t}}}^{\to\, x'(0)} + f_y(a,b)\overbrace{\boxed{\frac{y(t)-y(0)}{t}}}^{\to\, y'(0)}$$

$$+ \boxed{\frac{\beta(x(t), y(t))}{t}} \to 0$$

全微分可能.

$$\to f_x(a,b)\,x'(0) + f_y(a,b)\,y'(0)$$

$$\left|\frac{\beta(x(t),y(t))}{t}\right| \longrightarrow 0$$

$$= \boxed{\frac{|\beta(x(t),y(t))|}{\sqrt{(x(t)-a)^2+(y(t)-b)^2}}} \times \sqrt{\left(\frac{x(t)-a}{t}\right)^2+\left(\frac{y(t)-b}{t}\right)^2}$$

$$\to 0 \quad (t \to 0) \qquad\qquad \downarrow \ \sqrt{x'(0)^2+y'(0)^2}$$

$x(t), y(t)$: 微分可能 $\longrightarrow$ 連続

$x(t) \to x(0) = a$, $t \to 0$, $y(t) \to y(0) = b$.

I will cheat you here.

$\longrightarrow$ $\underline{(x(t), y(t)) \longrightarrow (a, b)}$

まとめ $F(t) = f(x(t), y(t))$

$$F'(t) = f_x(x(t), y(t))\,x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

$$\begin{cases} x + y - 1 = 0 \text{ の下で} \\ z = x^{\frac{1}{2}} y^{\frac{1}{2}} \end{cases}$$

$$\begin{cases} x + y - 1 = 0 \text{ の下で} \\ z = x^{\frac{1}{3}} y^{\frac{2}{3}} \end{cases}$$

Lagrange の未定乗数法　　CT 283p

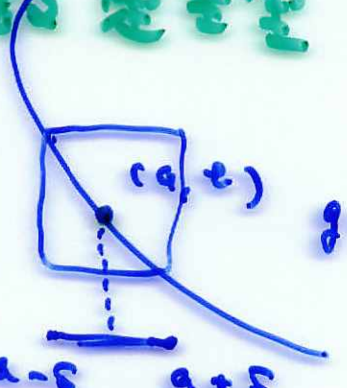
$$\begin{cases} g(x,y)=0 \text{ の下で} \\ z=f(x,y) \text{ の最大値・最小値} \end{cases}$$

定理 8.23　陰関数定理

g　C^1

$g(a,b)=0$

$g_y(a,b) \neq 0$



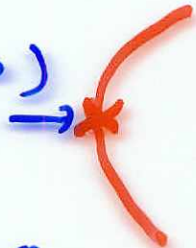
$g(x,y)=0$

$\Rightarrow$ (a,b) の近くで

$$\left(\begin{array}{l} g(x,y)=0 \\ \text{曲線上} \end{array} \iff y=\varphi(x) \right)$$

φ　C^1

をみたす $(a-\delta, a+\delta) \xrightarrow{\varphi} \mathbb{R}$ が存在する.



$(-1,0)$

$g_y=0$

$x^2+y^2=1$

$g=x^2+y^2-1$

$g_y=2y$

仮 (a,b) で極値をとる.

$g_y(a,b) \neq 0$

$g(a,b) = 0$

陰関数定理

$g(x,y)=0$

$\iff y = \varphi(x)$

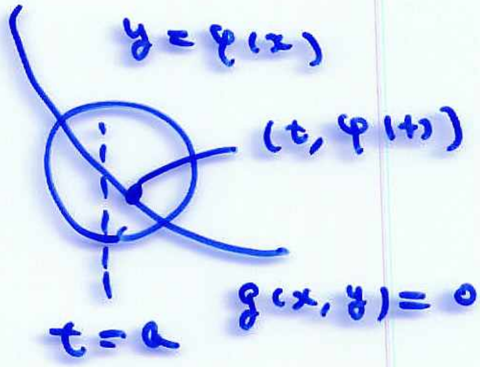
$x \sim a$

$\boxed{\begin{array}{c} g(x,y)=0 \text{ の下で} \\ z = f(x,y) \end{array}}$

例

$x^2+y^2-1=0$ の下で

$z = 3x+4y$



$$F(t) = f(t, \varphi(t))$$

$$F'(t) = f_x(t,\varphi(t)) \cdot 1 + f_y(t,\varphi(t)) \cdot \varphi'(t)$$

$\varphi(a) = b$

$$F'(a) = f_x(a,b) + f_y(a,b) \cdot \underset{?}{\underbrace{\varphi'(a)}} = 0$$

$$G(t) = g(t, \varphi(t)) \equiv 0$$

$$\underbrace{\boxed{g_x(t,\varphi(t)) \cdot 1 + g_y(t,\varphi(t)) \cdot \varphi'(t)}}_{= G'(t)} = 0$$

$t=a$ を代入

$$g_x(a,b) + \underset{\neq 0}{g_y(a,b)} \cdot \varphi'(a) = 0.$$

$$\leadsto \varphi'(a) = -\frac{g_x(a,b)}{g_y(a,b)}$$

$$\begin{cases} f_x(a,b) + f_y(a,b)\,\varphi'(a) = 0 \\ \varphi'(a) = -\dfrac{g_x(a,b)}{g_y(a,b)} \end{cases}$$

$$f_x(a,b) + f_y(a,b)\left(-\frac{g_x(a,b)}{g_y(a,b)}\right) = 0$$

$$\lambda = \frac{f_y(a,b)}{g_y(a,b)}$$

Lagrange の
未定乗数
multiplier

$$\begin{cases} f_x(a,b) = \lambda\, g_x(a,b) \\ f_y(a,b) = \lambda\, g_y(a,b) \end{cases}$$

条件付

定理 極値問題 subject to

$$z = f(x, y) \quad (\text{s.t. } g(x, y) = 0)$$

(a, b) で極値をとる

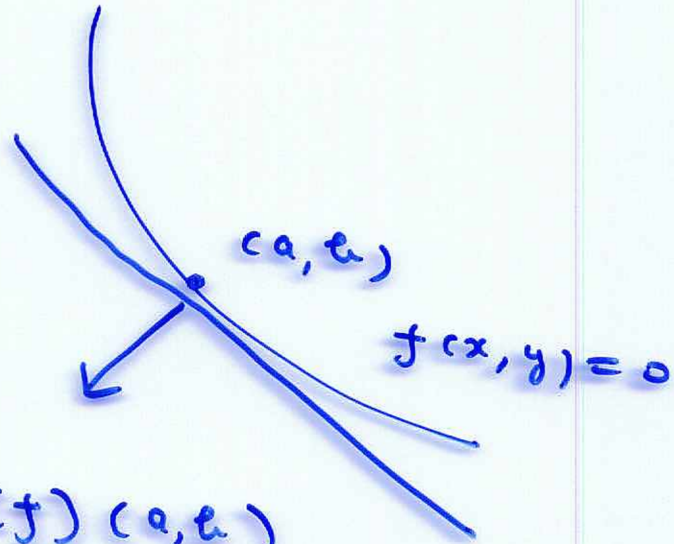
$$\Longrightarrow \begin{cases} f_x(a, b) = \lambda g_x(a, b) \\ f_y(a, b) = \lambda g_y(a, b) \end{cases}$$

をみたす λ がある

$$\begin{pmatrix} f_x(a, b) \\ f_y(a, b) \end{pmatrix} = \lambda \begin{pmatrix} g_x(a, b) \\ g_y(a, b) \end{pmatrix}$$

$$\nabla(f)(a, b) = \lambda \nabla(g)(a, b)$$

意味 2曲線 $f(x, y) = f(a, b)$ と $g(x, y) = 0$ は (a, b) で接している。

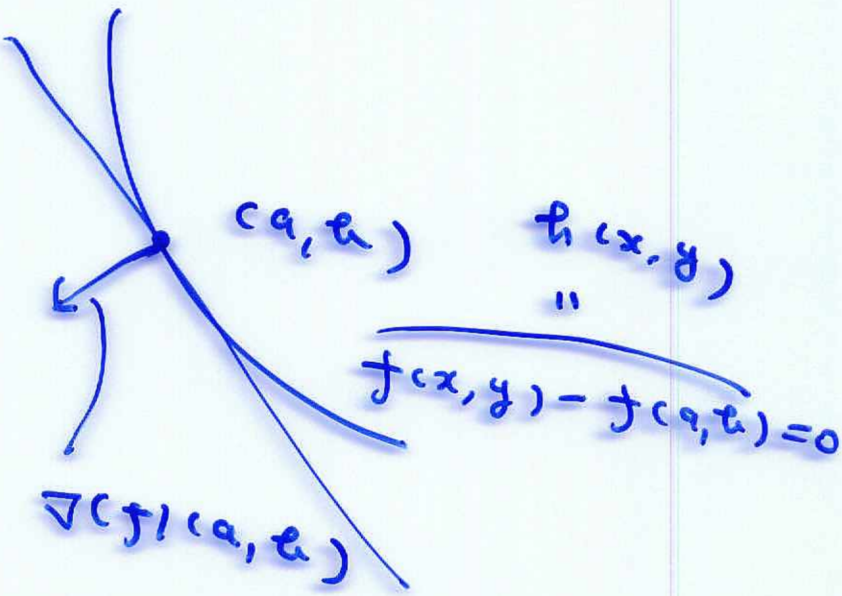


$$\nabla(f)(a, b) = \begin{pmatrix} f_x(a, b) \\ f_y(a, b) \end{pmatrix}$$

$h_x(x, y) = f_x(x, y)$

$h_y(x, y) = f_y(x, y)$

$\nabla(h)(a, b) = \nabla(f)(a, b)$

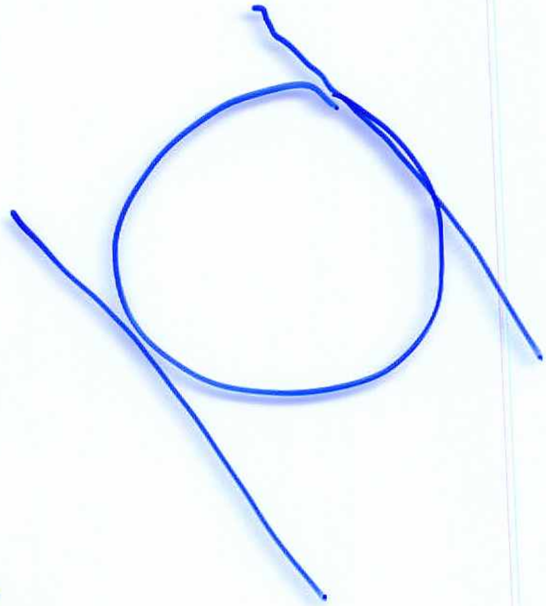


$g = x^2 + y^2 - 1 = 0$ の下で

$$z = f(x, y) = 3x + 4y.$$

$g_x = 2x, \quad g_y = 2y$

$f_x = 3, \quad f_y = 4$



$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$\rightsquigarrow \lambda \neq 0$

$$x = \frac{3}{2\lambda}, \quad y = \frac{4}{2\lambda}$$

$$x^2 + y^2 = \frac{9}{4\lambda^2} + \frac{16}{4\lambda^2}$$

$$= \frac{25}{4\lambda^2} = 1$$

$$\lambda = \pm \frac{5}{2}$$

$$(x, y) = \left(\pm \frac{3}{5}, \pm \frac{4}{5}\right)$$

両川点

で(のみ)極値をとる可能性あり).