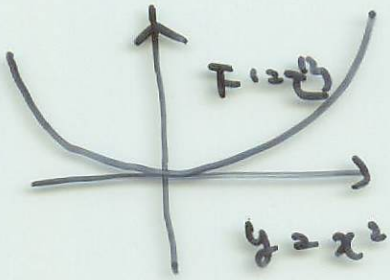
曲がり, 凹 凸

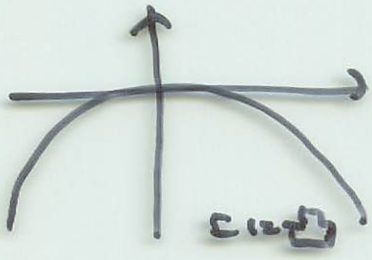
$y = f(x)$

(仮定)
$f''(x) > 0$ 常に

$<$



$y = -x^2$



傾き.

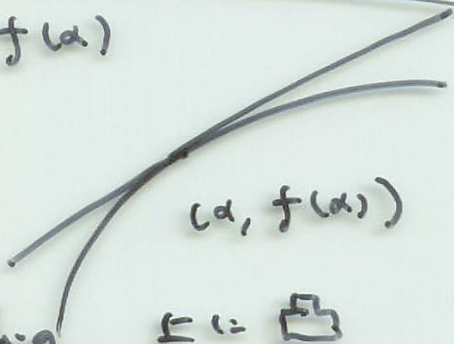
接線 $y = f'(\alpha)(t-\alpha) + f(\alpha) \longrightarrow (\alpha, f(\alpha))$ を通る.

$\Longrightarrow$ $t \neq \alpha$ ならば
$f(t) > f'(\alpha)(t-\alpha) + f(\alpha)$

$<$

$F(t) := f(t) - f'(\alpha)(t-\alpha) - f(\alpha)$

$F'(t) = f'(t) - f'(\alpha)\cdot 1$
$= f'(t) - f'(\alpha)$



$F''(t) > 0 \rightsquigarrow F'(t)$ は狭義の単調増加.

$s < \alpha < t$
$F'(s) < F'(\alpha) < F'(t)$
$\quad\quad\quad\; \shortparallel$
$f'(\alpha) - f'(\alpha) = 0$

t		α	
F'	$-$	0	$+$
F	$\searrow$	0	$\nearrow$

$t > \alpha \Longrightarrow F(t) > 0$
$t < \alpha \Longrightarrow F(t) > 0$

(1) $y = t\log t$ $(t > 0)$

$(fg)' = f'g + fg'$

$(\log t)' = \frac{1}{t}$

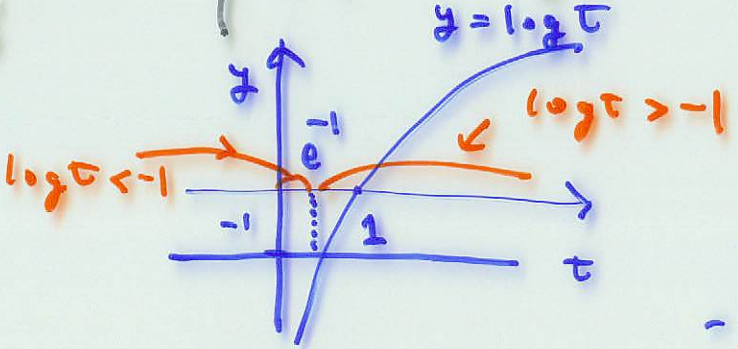
$y' = (t)'\log t + t(\log t)'$

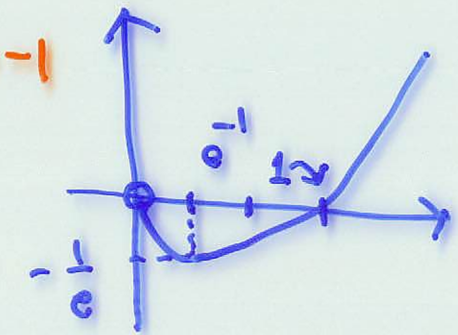
$= 1 \cdot \log t + t \cdot \frac{1}{t}$

$= \log t + 1$

$y' \gtreqless 0$

$\Longleftrightarrow t \gtreqless e^{-1}$





$\log t = -1$

$\Longleftrightarrow t = e^{-1}$

t	(0)		e^{-1}	
y'		$-$	0	$+$
y	(0)	$\searrow$	$-\frac{1}{e}$	$\nearrow$

$e^{-1} \cdot \log e^{-1} = -\frac{1}{e}$

Ⅰ $t \to +\infty$

$t\log t \to +\infty$

$\log t \to +\infty$
$(t \to +\infty)$

Ⅱ $t \to +0$ $\log t \to -\infty$

$t\log t \to 0$

いう>0'.

$y'' = \frac{1}{t} > 0$

下に凸

t	(0)		e^{-1}	
y'		$-$	0	$+$
y''		$+$	$+$	$+$
y	(0)	↳	$-\frac{1}{e}$	⤴

(2) $y = \dfrac{\log t}{t}$

$\left(\dfrac{g}{f}\right)' = \dfrac{g'f - gf'}{f^2}$

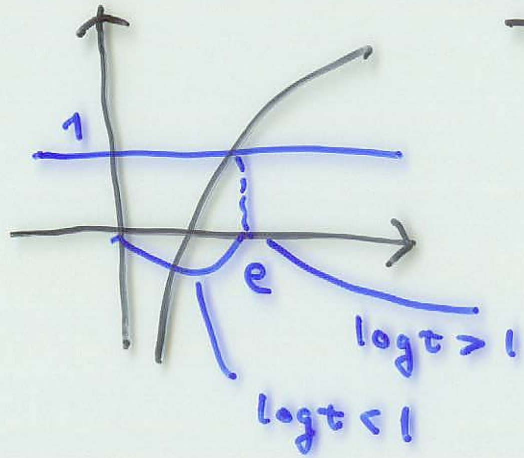
$y' = \dfrac{(\log t)' t - (t)' \log t}{t^2}$

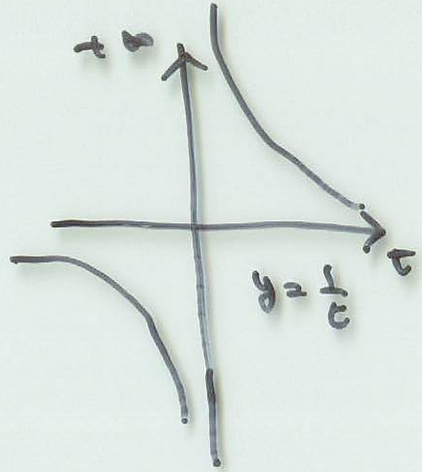
$(\log t)' = \dfrac{1}{t}$

$= \dfrac{\frac{1}{t} \cdot t - 1 \cdot \log t}{t^2}$

$= \dfrac{1 - \log t}{t^2}$

$t^2 > 0$





$y' \gtreqless 0 \iff \log t \lesseqgtr 1$

$\iff t \lesseqgtr e$

t	(0)		e	
y'		$+$	0	$-$
y	$(-\infty)$	$\nearrow$	$\frac{1}{e}$	$\searrow$

I $t \to +0$

$\left.\begin{array}{l} \dfrac{1}{t} \to +\infty \\ \log t \to -\infty \end{array}\right\}$

$\dfrac{\log t}{t} \to -\infty$

II $\dfrac{\log t}{t} \to 0 \quad (t \to +\infty)$

後述.

GNUPLOT

(3) $y = \sqrt{x}\,(x-1)$ $\quad (x > 0)$

$(\sqrt{x})' = \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$

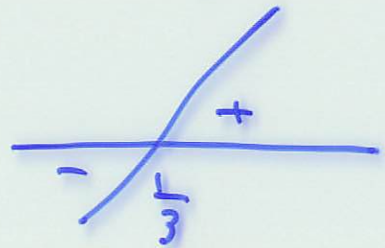
$$y' = (\sqrt{x})'(x-1) + \sqrt{x}\,(x-1)'$$

$$= \frac{1}{2}\frac{1}{\sqrt{x}}(x-1) + \sqrt{x} \cdot 1$$

$$= \frac{(x-1) + \sqrt{x} \cdot 2\sqrt{x}}{2\sqrt{x}}$$

$$= \frac{3x-1}{2\sqrt{x}}$$

$2\sqrt{x} > 0$

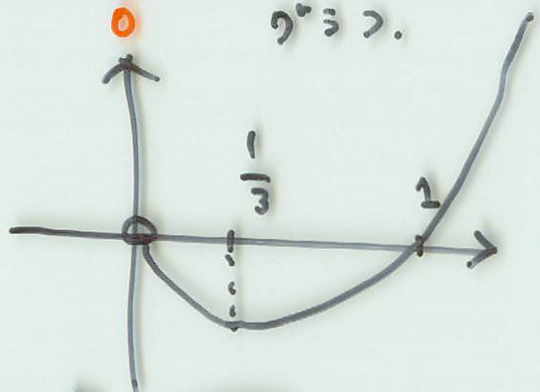


x	(0)		$\frac{1}{3}$	
y'		$-$	0	$+$
y	(0)	↘		↗

$y = \sqrt{x}\,(x-1)$

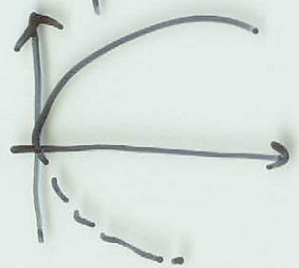
$\sqrt{x} > 0$

グラフ.



Ⓘ $x \to +0 \quad \sqrt{x} \to 0$

$\sqrt{x}\,(x-1) \to 0 \times (0-1)$
$= 0$



(5) $y = t^2 e^{-t}$ $\qquad (e^{ct})' = c\,e^{ct}$

$y' = 2t\,e^{-t} + t^2(-e^{-t})$

$= e^{-t}(2t - t^2)$

$= \underset{\substack{\vee \\ 0}}{e^{-t}}\, t(2-t)$

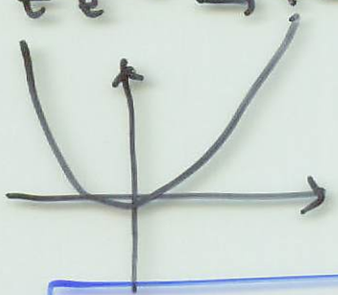
t		0		2	
y'	−	0	+	0	−
y	↘	0	↗	$\frac{4}{e^2}$	↘

Ⓘ $t \to -\infty$

$-t \to +\infty$

$\hookrightarrow e^{-t} \to +\infty$

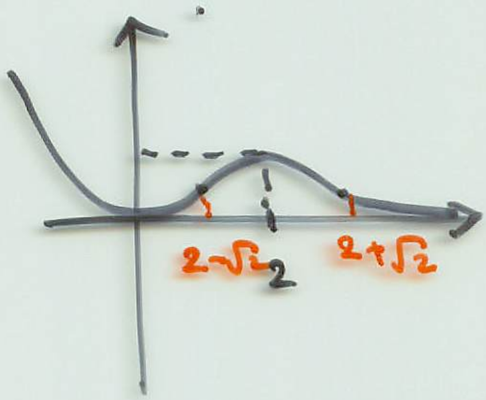
$t^2 e^{-t} \to +\infty$



Ⓘⓘ $t \to +\infty$

$y = \dfrac{t^2}{e^t} \longrightarrow 0$

$$\boxed{R > 1 \qquad \frac{n^k}{R^n} \longrightarrow 0}$$



$y = t^2 e^{-t}$

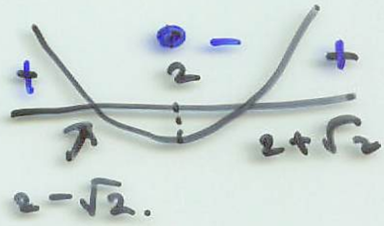
$y' = e^{-t}(2t - t^2)$

$y'' = -e^{-t}(2t - t^2) + e^{-t}\cdot(2 - 2t)$

$= e^{-t}(t^2 - 2t - 2t + 2)$

$= \underset{\substack{\vee \\ 0}}{e^{-t}}(t^2 - 4t + 2)$

$$t^2-4t+2=(t-2)^2-4+2$$
$$=(t-2)^2-2.$$



$$\left.\begin{aligned}(t-2)^2&=2\\ t-2&=\pm\sqrt{2}\\ t&=2\pm\sqrt{2}\end{aligned}\right\}$$

N.B.
$0<2-\sqrt{2}<2<2+\sqrt{2}$

t		0		$2-\sqrt{2}$		2		$2+\sqrt{2}$	
y'	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$
y''	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
y	↘	0	↗		↗		↘		↘

・Taylor 展開.

微分方程式 ⟶ 経済数学

$$\left(\quad y'(t)=\alpha y(t)\longrightarrow y(t)=y(0)e^{\alpha t}+\alpha\right.$$

・積分

・2変数の偏微分 ⟶ 経済数学.

連続性 Ⅰ 極限

$c > 0$ 　　　　　　　　← または

$$f : (a-c,\ a) \cup (a,\ a+c) \longrightarrow \mathbb{R}$$

$$\boxed{\begin{array}{l} x_n \in (a-c,\ a+c) \\ x_n \neq a,\ x_n \to a \end{array}}$$

$f(x) \longrightarrow A \quad (x \to a)$

$\Longleftrightarrow$ との性質をみたす $\{x_n\}$ に対して

$$f(x_n) \longrightarrow A$$

$$f : (a_1,\ a_2) \longrightarrow \mathbb{R}$$

Ⅱ　$f(x) \longrightarrow f(a) \quad (x \to a)$

$\Longleftrightarrow$ f は $x = a$ で連続

$f(a) > \overset{c}{0}$ とする

＃
$$\boxed{\begin{array}{l} \delta > 0 \text{ があって} \\ \quad x \in (a-\delta,\ a+\delta) \\ \qquad \Longrightarrow f(x) > 0 \end{array}}$$

背理法で．＃ を否定．

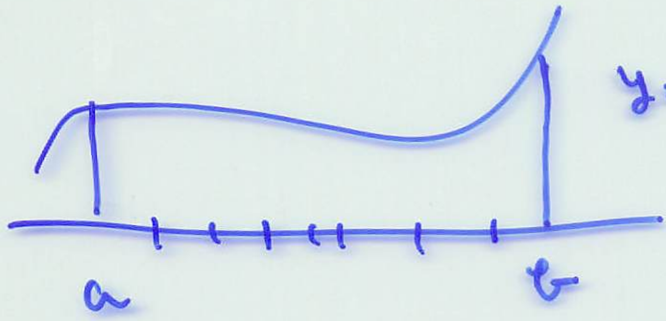
$$\left(\begin{array}{l} \text{任意の } \delta > 0 \text{ に対して} \\ \quad x \in (a-\delta,\ a+\delta) \text{ かつ } f(x) \leqq 0 \\ \text{をみたす } x \text{ がある.} \end{array}\right.$$

$\delta = \frac{1}{n}$ とする．　$\left.\begin{array}{l} \underset{\downarrow\, a}{a-\frac{1}{n}} < \underset{\downarrow\, a}{x_n} < \underset{\downarrow\, a}{a+\frac{1}{n}} \\ f(x_n) \leqq 0. \end{array}\right)$ をみたす x_n がある．

「はさみうち」から　$x_n \to a$.

$f(x_n) \leqq 0 \rightsquigarrow f(a) \leqq 0.$ 矛盾．

$n \longrightarrow +\infty$ 　$\llcorner\!\!\longrightarrow f(a)$



$y=f(x)$ 連続.

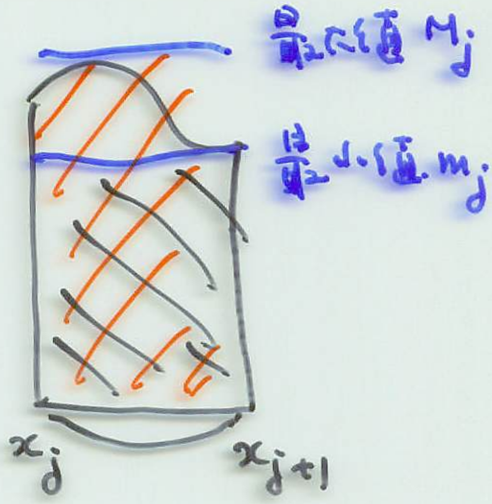
$\Delta : a = x_0 < x_1 < x_2 < \cdots < x_n = b$

区間 $[a, b]$ の分割

$I_j := [x_j, x_{j+1}]$

$M_j = \max_{I_j \ni t} f(t)$

$m_j = \min_{I_j \ni t} f(t)$



$$\left.\begin{aligned} S_\Delta &= \sum_{j=0}^{n-1} M_j (x_{j+1} - x_j) \\ s_\Delta &= \sum_j m_j (x_{j+1} - x_j) \end{aligned}\right\}$$

$s_\Delta \le S_\Delta$

任意の j に対して

$m_j \le M_j$

$|\Delta| = \max_j (x_{j+1} - x_j)$

分割の細かさ

<u>定理</u>

$|\Delta| \to +0$ のとき

$$S_\Delta ,\ s_\Delta \longrightarrow S = \int_a^b f(t)\,dt.$$

分割の列. $\{\Delta_\ell\}$ $|\Delta_\ell| \to +0$ のとき.

$S_{\Delta_\ell} \longrightarrow S$ ← $\{\Delta_\ell\}$ の列の取り方に依らず...

$$m_j \leq f(x_j) \leq M_j$$

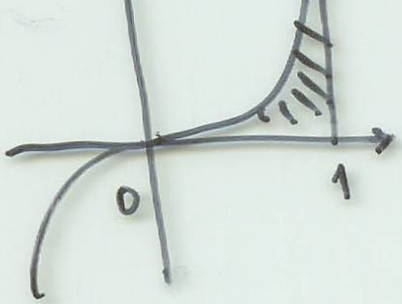
$$\sum_{j=1}^{n} m_j \frac{1}{n} \leq \sum_{j=0}^{n-1} f(x_j)\cdot\frac{1}{n} \leq \sum_{j=0}^{n-1} M_j \cdot \frac{1}{n}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\int_a^b f(t)\,dt \qquad \int_a^b f(t)\,dt. \qquad \int_a^b f(t)\,dt.$$

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\underbrace{a+\frac{k}{n}(b-a)}_{\substack{\Downarrow \\ x_k}}) \longrightarrow \int_a^b f(t)\,dt$$

$$\frac{1}{n}\sum_{k=1}^{n} f\left(a+\frac{k}{n}(b-a)\right) \nearrow$$



$$\sum_{k=1}^{n}\frac{1}{n}\cdot\left(\frac{k}{n}\right)^3 = \frac{1}{n^4}\sum_{k=1}^{n} k^3$$

$$= \frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4} \qquad \boxed{\left(\frac{1}{4}x^4\right)' = x^3}$$

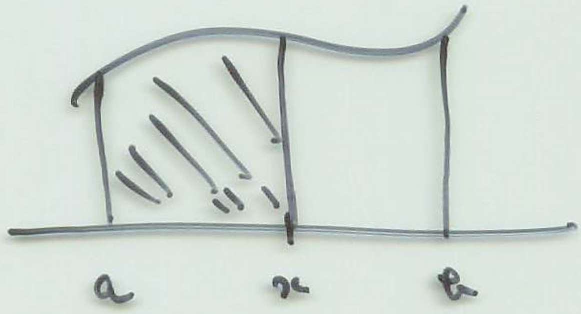
$$= \frac{1}{4}\cdot\left(1+\frac{1}{n}\right)^2 \longrightarrow \frac{1}{4}$$

$$\uparrow$$

$f(x) = x^3$ $\qquad\qquad \int_0^1 x^3\,dx.$

$$g(x) = \frac{1}{4}x^4 = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}\cdot 1 - \frac{1}{4}\cdot 0$$

台形公式.



微分積分学の基本定理.

<u>定理</u>: $f : [a, b] \longrightarrow \mathbb{R}$
各点で連続.

$$F(x) = \int_a^x f(t)\,dt \qquad (a \leqq x \leqq b)$$

$$\Longrightarrow \quad F'(x) = f(x).$$

f の
G : 不定積分.
f の原始関数.

系 $G'(x) = f(x)$ とする.

$$\int_a^b f(x)\,dx = G(b) - G(a)$$
$$= \left[G \right]_a^b$$

(証明) $(G - F)' = G' - F' = f - f \equiv 0.$

$\rightsquigarrow G(x) - F(x) = C$ 定数 $\quad F(a) = \int_a^a f(t)\,dt = 0$

$$\underset{\substack{\shortparallel \\ 0}}{G(a) - F(a)} = C \rightsquigarrow C = G(a)$$

$$G(b) - F(b) = C \rightsquigarrow \underset{\substack{\shortparallel \\ \int_a^b f(t)\,dt}}{F(b)} = G(b) - C$$
$$= G(b) - G(a)$$

$G'(x) = f(x)$ のとき $\qquad G(x) = \int f(x)\,dx + C$

① $(G(x) + C)' = G' + 0 = G' = f.$

② $G_1' = f,\ G_2' = f$

$\rightarrow (G_1 - G_2)' = f' - f' \equiv 0$

$\rightarrow \quad G_1 - G_2 = $ 定数 $= C$

$G_1 = G_2 + C$

---

① $(x^{n+1})' = (n+1)x^n$

$\left(\frac{1}{n+1} x^{n+1}\right)' = x^n \qquad \int x^n\,dx = \frac{1}{n+1} x^{n+1} + C$

② $(e^t)' = e^t \qquad \int e^t\,dt = e^t + C$

③ $(t^{\alpha+1})' = (\alpha+1)t^\alpha$

$\alpha \neq -1 \quad \left(\frac{1}{\alpha+1} t^{\alpha+1}\right)' = t^\alpha \qquad \int t^\alpha\,dt = \frac{t^{\alpha+1}}{\alpha+1} + C$

$$\int \sqrt{t}\,dt = \frac{1}{1+\frac{1}{2}} t^{\frac{3}{2}} + C$$

$F'(x) = f(x). \iff F(x) = \int f(x)\,dx + C$

$G'(x) = g(x) \iff G(x) = \int g(x)\,dx + C$

$\rightarrow (F(x) \pm G(x))' = f(x) \pm g(x).$

$\int (f(x) \pm g(x))\,dt = \int f(x)\,dx \pm \int g(x)\,dx.$

$(cF)' = cF' = cf.$

$$\int c f(x)\,dx = c\int f(x)\,dx$$

$$\int (x^3 - 2x + 5)\,dx$$
$$= \frac{x^4}{4} - 2\cdot\frac{x^2}{2} + 5x + C.$$

部分積分, 置換積分.

高校2年.

$\boxed{\begin{array}{l}\text{微分 3次式}\\ \text{積分 2次式.}\end{array}}$

$F' = f$
$G' = g.$

$$(FG)' = F'G + FG'$$
$$= fG + Fg.$$

$$FG = \int f(x)G(x)\,dx + \int F(x)\,g(x)\,dx$$

$$\boxed{\int f(x)G(x)\,dx = FG - \int F(x)\,g(x)\,dx.}$$

部分積分 $F = e^t,\ G = t$

例 $$\int t e^t\,dt = \int (e^t)' t\,dt = e^t\cdot t - \int e^t\cdot 1\,dt$$
$$= e^t t - e^t + C$$

$$\int \log t\, dt = \int \overset{f}{(t)'}\, \overset{G}{\log t}\, dt$$

$$= t\log t - \int t\cdot \overset{\frac{1}{t}}{\overset{\|}{(\log t)'}}\, dt$$

$G = \log t$ $\quad = t\log t - \int 1\, dt$

$F = t.$ $\quad = t\log t - t + C$

積分定数.

$$\int t^2 e^t\, dt = \int t^2\cdot(e^t)'\, dt$$

$$= t^2 e^t - \int 2t\cdot e^t\, dt$$

$$= t^2 e^t - 2\int t(e^t)'\, dt$$

$$= t^2 e^t - 2te^t + 2\int \overset{1}{\overset{\|}{(t)'}}\, e^t\, dt$$

$$= t^2 e^t - 2te^t + 2e^t + C'$$

$$\int t\log t\, dt = \int \left(\frac{t^2}{2}\right)' \log t\, dt$$

$$= \frac{t^2}{2}\log t - \int \frac{t^2}{2}\cdot \overset{\frac{1}{t}}{\overset{\|}{(\log t)'}}\, dt$$

$$= \frac{t^2}{2}\log t - \int \frac{t}{2}\, dt$$

$$= \frac{t^2}{2}\log t - \frac{1}{2}\cdot\frac{t^2}{2} + C$$

$$= \frac{t^2}{2}\log t - \frac{1}{4}t^2 + C'.$$

定理 f が $x=a$ で連続とする

$f(a) > c$ ($< c$) $\Rightarrow$ $\delta > 0$ があって

[$(a-\delta, a+\delta) \ni x$

$\Rightarrow f(x) > c$ ($< c$)

→ 微分積分学の基本定理.

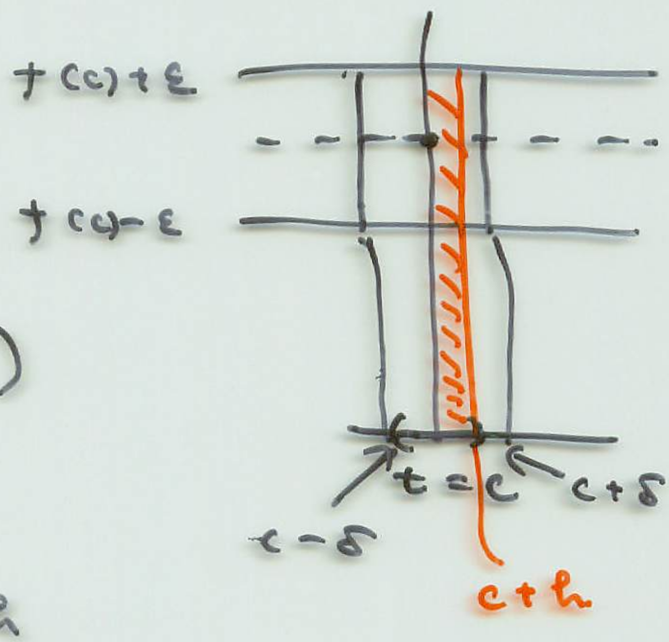
f は $x=c$ で連続.

任意の $\varepsilon > 0$ をとる. $f(c)-\varepsilon < f(c) < f(c)+\varepsilon$

$\delta > 0$ があって

$f(c)-\varepsilon < f(t) < f(c)+\varepsilon$

$(c-\delta < t < c+\delta)$



$0 < h < \delta$

$$\frac{1}{h}\times h\,(f(c)-\varepsilon) < \frac{1}{h}\int_c^{c+h} f(t)\,dt < \frac{1}{h}\times h \times (f(c)+\varepsilon)$$

$$f(c)-\varepsilon < \frac{1}{h}\int_c^{c+h} f(t)\,dt < f(c)+\varepsilon$$

$y = x^3 - 3x + 1$

$y' = 3x^2 - 3 = 3(x^2 - 1)$

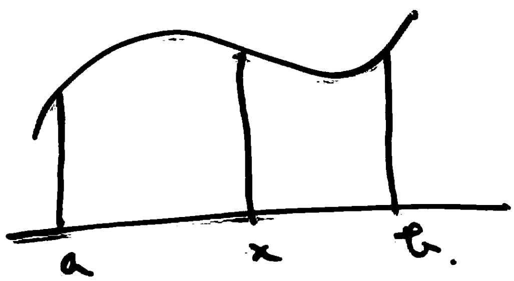
2와 1사이값.

$$\int f(x)G(x)\,dx = F(x)G(x) - \int F(x)g(x)\,dx$$

$$\boxed{\int F'G\,dx = F(x)G(x) - \int FG'\,dx}$$

$$\int_a^b F'G\,dx = \Big[F(x)G(x)\Big]_a^b - \int_a^b FG'\,dx.$$

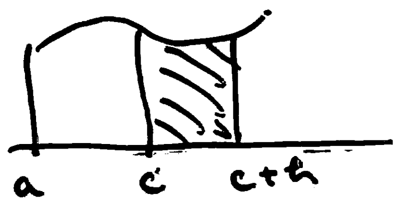
微分積分学の基本定理.



$$F(x) = \int_a^x f(t)\,dt$$

$h>0$.

$$\frac{F(c+h) - F(c)}{h}$$



$$= \frac{1}{h}\int_c^{c+h} f(t)\,dt.$$

$$(e^{ct})' = c\,e^{ct}$$

(1) $\displaystyle\int_0^1 e^{-2t}\,dt.$

(3) $\displaystyle\int_1^e t\log t\,dt$

(2) $\displaystyle\int_0^1 t\,e^{-t}\,dt$

(4) $\displaystyle\int_1^e t^2\log t\,dt.$

明日　置換積分, Taylor展開