2項定理

04年8月2日 (2回目)

$$(x+y) = x+y$$

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

$$(x+y)^4 x = x^5+4x^4y+6x^3y^2+4x^2y^3+xy^4$$

$$+)\ (x+y)^4 y = x^4y+4x^3y^2+6x^2y^3+4xy^4+y^5$$

$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

⋮

Pascalの三角形

$$\boxed{\begin{aligned}(x+y)^n &= x^n + {}_nC_1 x^{n-1}y + {}_nC_2 x^{n-2}y^2 + \cdots \\ &\quad \cdots + {}_nC_{n-2} x^2 y^{n-2} \\ &\quad + {}_nC_{n-1} x y^{n-1} + y^n\end{aligned}}$$

2項展開.

$$= \sum_{k=0}^{n} {}_nC_k\, x^{n-k} y^k$$

$_nC_k$ n個の物から k個選ぶ場合の数.

① ② ③ ---- ⓝ

k個を選んで並べる.

1番目 2番目 --- k番目

○ ○ --- ○

$$= \frac{n \times (n-1) \times \cdots \times (n-k+1) \times (n-k) \times \cdots \times 1}{k \times (k-1) \times \cdots \times 1 \times (n-k) \times \cdots \times 1}$$

$$= \frac{n!}{k!\,(n-k)!}$$

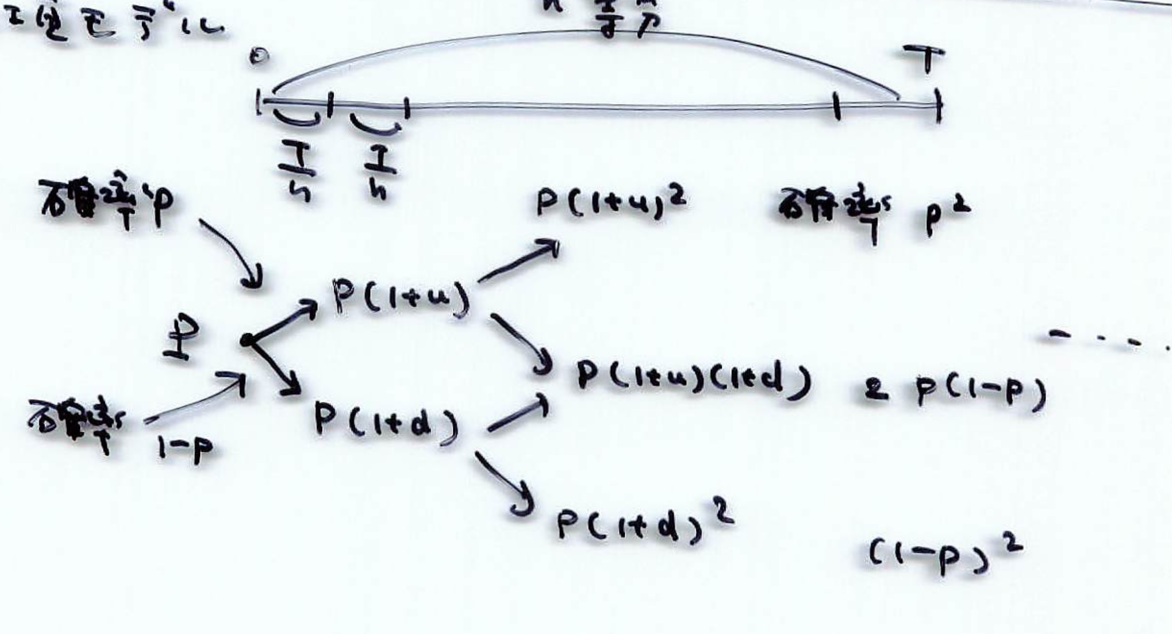
因子 = factor

2項定理

$$\underbrace{(x+y)(x+y)\cdots\cdots(x+y)}_{n\ 因子.}$$

$x^{n-k}\,y^{k}$ は $_nC_k$ 回でてくる.

2項モデル



⟶ ブラック・ショールズ

数列の収束

$a_1, a_2, a_3, \cdots\cdots$

→ 面積 $S_n = \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$n = 1, 2, 3, \cdots\cdots$

$a_n \longrightarrow \alpha \in \mathbb{R}$　　$\mathbb{R}$: 実数の集合

属する

element 元, 要素.

→ $\in$　ε

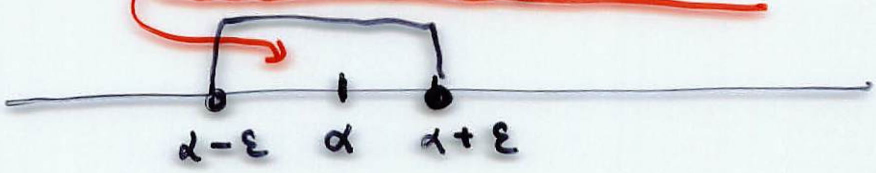
$\Longleftrightarrow$ 任意に正数 $\varepsilon > 0$ を取る.

定義

ある番号 N をとって　　$\varepsilon > 0$ に依る.

$$|a_n - \alpha| < \varepsilon \quad (n = N, N+1, N+2, \cdots)$$

$a_N, a_{N+1}, a_{N+2}, \cdots\cdots$



~~定理~~　例 ①

$a_n = c$　　　　$a_n \longrightarrow c$

$c, c, c, c, \cdots\cdots$ ($c = a_1$)

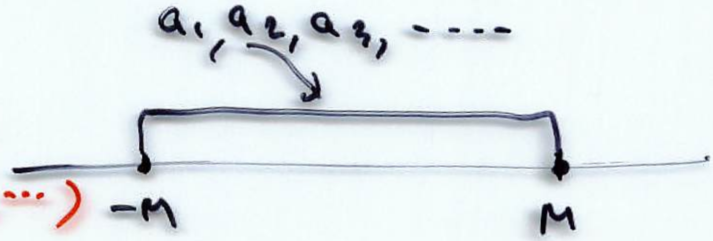
② $a_n = \frac{1}{n}$ $(n = 1, 2, 3, \cdots\cdots)$

$a_n \longrightarrow 0 \quad (n \longrightarrow +\infty)$

定理 $a_n \to \alpha \in \mathbb{R}$. $(n \to +\infty)$ とする.

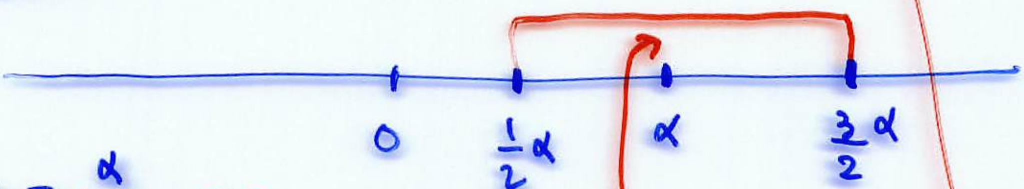
$\{a_n\}$ は有界である. 即ち ある正数 $M > 0$ に対して

$|a_n| \le M$ $(n = 1, 2, \cdots)$



idea

$\alpha > 0$



$\varepsilon = \frac{\alpha}{2}$ あるNに対して

$a_N, a_{N+1}, a_{N+2}, \cdots$

$a_1, a_2, \cdots, a_{N-1}$?

有限個

$|a_1|, |a_2|, \cdots, |a_{N-1}|, \frac{3}{2}\alpha$ の最大値 M.

定理 $a_n \to A$, $b_n \to B$ $(n \to +\infty)$

(i) $a_n \pm b_n \longrightarrow A \pm B$.

(ii) $a_n \cdot b_n \longrightarrow AB$.

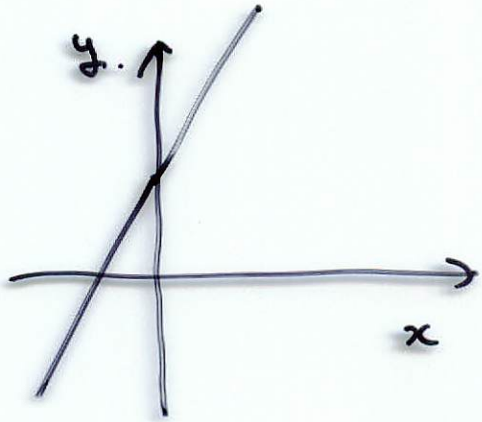
(iii) $a_n \neq 0$ (全ての n), $A \neq 0$

$\frac{1}{a_n} \longrightarrow \frac{1}{A}$.

定理 (はさみうち) $a_n \to A$, $b_n \to A$ $(n \to +\infty)$

$a_n \le c_n \le b_n$ (全ての n)

$\Longrightarrow$ $c_n \to A$ $(n \to +\infty)$



1) $y = 3x + 1$

$$\frac{(3x+1)-(3\alpha+1)}{x-\alpha}$$

$$= \frac{3(x-\alpha)}{x-\alpha} = 3.$$

直線 $y = ax + b$

$$\frac{\Delta x}{\Delta y} = a.$$

2) $y = 5$

$$\frac{5-5}{x-\alpha} = 0 \to 0$$

$(x \to \alpha)$

---

3)

$f(x) = Ax^2 + Bx + C$

$$\frac{(Ax^2 + Bx + C) - (A\alpha^2 + B\alpha + C)}{x-\alpha}$$

$$= A\,\frac{x^2-\alpha}{x-\alpha} + B\,\frac{x-\alpha}{x-\alpha}$$

$$= A(x+\alpha) + B.$$

$x \to \alpha$

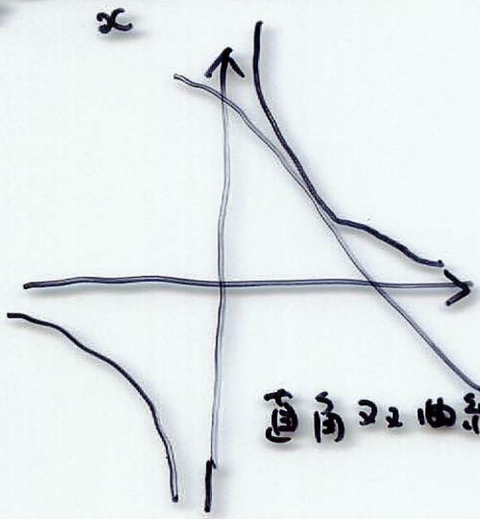
$x_n \to \alpha$

$$= Ax_n + A\alpha + B$$

$$\longrightarrow A\alpha + A\alpha + B = 2A\alpha + B.$$

$$f'(\alpha) = 2A\alpha + B.$$

$y = \dfrac{1}{x} \quad (x \neq 0)$



直角双曲線.

$x \to \alpha$ のとき

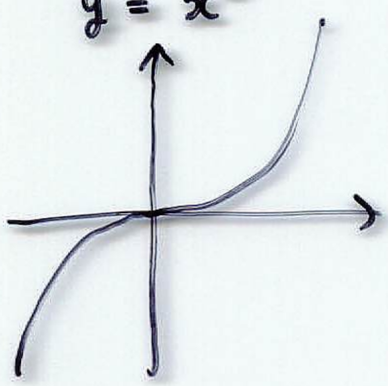
$$\frac{\frac{1}{x} - \frac{1}{\alpha}}{x - \alpha} = \frac{\alpha - x}{x \cdot \alpha (x - \alpha)}$$

$$= -\frac{1}{x} \cdot \frac{1}{\alpha} \longrightarrow -\frac{1}{\alpha^2}$$

$$\left(\begin{array}{l} -\dfrac{1}{x_n} \cdot \dfrac{1}{\alpha} \longrightarrow -\dfrac{1}{\alpha} \cdot \dfrac{1}{\alpha} \\ x_n \longrightarrow \alpha \neq 0 \end{array} \right)$$

演習 $y = \dfrac{1}{x^2}$ の $x = \alpha$ ($\alpha \neq 0$) における微分係数を求めよ.

$y = x^3$



$$\frac{x^3 - \alpha^3}{x - \alpha} = \frac{\cancel{(x-\alpha)}(x^2 + \alpha x + \alpha^2)}{\cancel{x - \alpha}}$$

$$= x^2 + \alpha x + \alpha^2.$$

$$\boxed{x^3 - \alpha^3 = (x - \alpha)(x^2 + \alpha x + \alpha^2)}$$

$x \to \alpha$ $\longrightarrow$ $\alpha \cdot \alpha + \alpha \cdot \alpha + \alpha^2$

$= 3\alpha^2$

導関数. $f(x) = x^3$ $\quad f'(\alpha) = 3\alpha^2$

$\alpha \rightsquigarrow x$ $\quad f'(x) = 3x^2.$

$(x^3)' = 3x^2.$

導関数.

$$S_n = \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \longrightarrow \frac{1}{6}\times 1\times 2 = \frac{1}{3}.$$

$$\begin{cases} 1+\frac{1}{n} \longrightarrow 1+0=1 \\ 2+\frac{1}{n} \longrightarrow 2+0=2 \end{cases}$$

(3:4

$$\frac{1}{n^2} = \frac{1}{n}\times\frac{1}{n} \longrightarrow 0\times 0 = 0$$

$$\frac{1}{n^3} = \frac{1}{n^2}\times\frac{1}{n} \longrightarrow 0\times 0 = 0$$

$$\frac{1}{n^k} \longrightarrow 0 \qquad (k=1,2,3,\cdots)$$

$$\frac{1}{3+n} = \frac{\frac{1}{n}}{\frac{3}{n}+1} \longrightarrow \frac{0}{3\times 0+1} = 0$$

$$\frac{4-n}{5+n} = \frac{\frac{4}{n}-1}{\frac{5}{n}+1} \longrightarrow \frac{0-1}{0+1} = -1.$$

$$\frac{2+3n}{1+n^2} = \frac{\frac{2}{n^2}+\frac{3}{n}}{\frac{1}{n^2}+1} \longrightarrow \frac{0+0}{0+1} = 0$$

Ex 0

(1) $a_n = \dfrac{3+n}{1+2n}$ (2) $b_n = \dfrac{2-n+3n^2}{1+n+n^2}$

$|r| < 1 \qquad r^n \longrightarrow 0 \ (n \to +\infty)$

I) $r > 0$ の場合示す. $\quad 0 < r < 1$

$s = \frac{1}{r} > 1 \rightsquigarrow s = 1 + \theta$ とおくと $\theta > 0$.

$$\frac{1}{r^n} = s^n = (1+\theta)^n = \boxed{1} + {}_nC_1\theta + \boxed{{}_nC_2\theta^2 + \cdots + \theta^n}$$

$\boxed{1} > 0$, ${}_nC_1 = n$, $\boxed{{}_nC_2\theta^2 + \cdots + \theta^n} > 0$

${}_nC_k > 0$

${}_nC_2 = \frac{n(n-1)}{2}$

$> n\theta$

$$0 < r^n < \frac{1}{n\theta} = \frac{1}{\theta} \cdot \frac{1}{n}$$

$0 \to 0$, $\frac{1}{n\theta} \to 0$

$r^n \to 0$ by はさみうち.

II) $r = 0$ 明らか. $0, 0, 0, ----$

III) $r \neq 0 \qquad s = |r|$ とおく. $\boxed{0 < s < 1}$

I) が使える

$$-s^n \leqq r^n \leqq s^n$$

$-s^n \to 0$, $r^n \to 0$, $s^n \to 0$

by はさみうち.

$|r| < 1$ のとき $\quad n r^n \longrightarrow 0 \ (n \to +\infty)$

To be explained.

レポートにしてもよい…

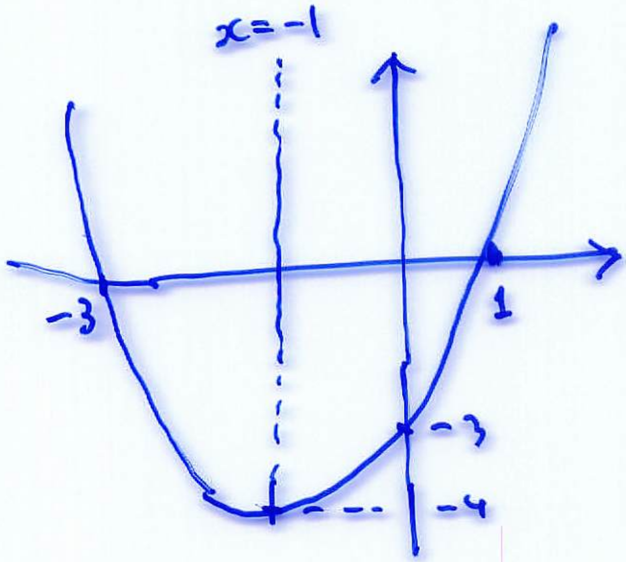
(答) (1) $y = \underset{\sim\sim\sim}{x^2 + 2x} - 3$

$= (x+1)^2 - 1^2 - 3$

$= (x+1)^2 - 4$

$(x+A)^2$

$= x^2 + 2Ax + \underset{\sim}{A^2}$



(2) $y = x^2 - 8x + 2 = (x-4)^2 - 16 + 2$

$= (x-4)^2 - 14.$

$x^2 - 8x + 2 = 0 \Longleftrightarrow (x-4)^2 = 14.$

$\Longleftrightarrow x - 4 = \pm\sqrt{14}$

$\Longleftrightarrow x = 4 \pm \sqrt{14}.$

$a_{n+1} = 3a_n - 2.$

$\lambda = 3\lambda - 2$

$\lambda = 1$

$$
\begin{array}{rl}
 & a_{n+1} = 3a_n - 2 \\
-\big) & 1 = 3 \times 1 - 2 \\
\hline
 & a_{n+1} - 1 = 3(a_n - 1)
\end{array}
$$

$a_n = 3^n(a_0 - 1) + 1$

$a_n - 1 = 3^n(a_0 - 1)$

$\cancel{(2)\ a_{n+1} = 3a_n - 2. \qquad \lambda = 3\lambda - 2}$

(2) $a_{n+1} = -2a_n + 1 \qquad \lambda = -2\lambda + 1$

$-)\ \frac{1}{3} = -2\times\frac{1}{3} + 1 \qquad 3\lambda = 1 \quad \lambda = \frac{1}{3}$

$a_{n+1} - \frac{1}{3} = -2\left(a_n - \frac{1}{3}\right)$

$a_n - \frac{1}{3} = (-2)^n\left(a_0 - \frac{1}{3}\right)$

$a_n = \frac{1}{3} + (-2)^n\left(a_0 - \frac{1}{3}\right)$

$$\boxed{|r|<1 \Rightarrow \sum_{n=0}^{+\infty} r^n = \frac{1}{1-r}}$$

$S_n = \sum_{k=0}^{n} a_k$ （第 n 部分和.）

$$S_n \to S \ (n\to+\infty) \text{ のとき } \sum_{n=0}^{+\infty} a_n = S.$$

$|r|<1$ のとき.

$$1+r+r^2+\cdots = \frac{1}{1-r}$$

$$S_n = 1+r+r^2+\cdots+r^n$$

$$= \frac{1-r^{n+1}}{1-r} = \frac{1}{1-r} - \frac{r}{1-r}\times r^n$$

$$\to \frac{1}{1-r} - \frac{r}{1-r}\times 0$$

$$= \frac{1}{1-r}$$

信用創造　x：準備金比率.

$$\text{預金 } d \to \begin{cases} xd & \text{準備金} \\ (1-x)d & \text{貸出し.} \end{cases}$$

$(1-x)d \to$ 還流して預金

貸出し. $\to (1-x)xd$ 準

貸出し. $\to (1-x)^2 d$

$\vdots$

創造された預金

$$d+(1-x)d+(1-x)^2d+(1-x)^3d+\cdots$$

$$= d\sum_{k=0}^{+\infty}(1-x)^k = d\,\frac{1}{1-(1-x)} = \frac{d}{x}.$$

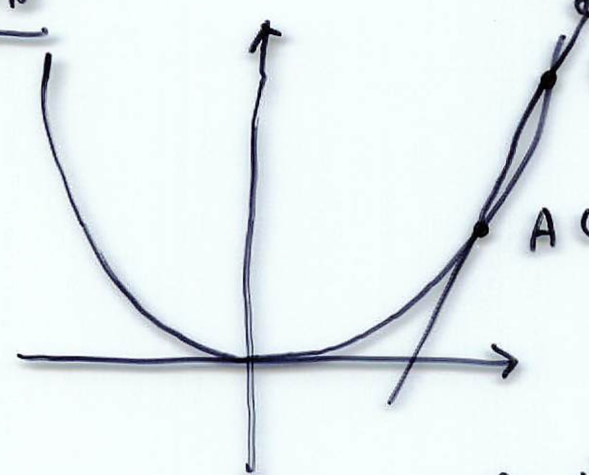
貸し出し総額.

$$(1-x)d + (1-x)^2 d + (1-x)^3 d + \cdots$$

$$= \underbrace{\sum_{k=0}^{+\infty} (1-x)^k}_{= \frac{1}{x}} (1-x)d = \boxed{\frac{1-x}{x} d}.$$

預金dが創造する信用.

微分



$x \neq \alpha$

$A \neq P$.

$$\text{直線APの傾き} = \frac{x^2-\alpha^2}{x-\alpha} = \text{平均変化率}.$$

$$\boxed{x^2-\alpha^2 = (x-\alpha)(x+\alpha)}$$

$$= \frac{(x-\alpha)(x+\alpha)}{x-\alpha} = x+\alpha.$$

$P \to A$

$(x \to \alpha) \longrightarrow 2\alpha$

$$x = \alpha + 10^{-N} \qquad x+\alpha = 2\alpha + 10^{-N}$$

Aにおける接線の傾き.

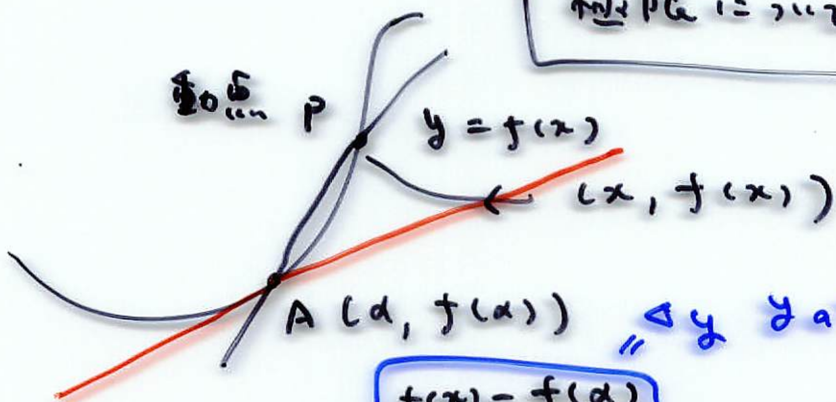
$$f(x) \longrightarrow A \quad (x \to a)$$

$\iff$ $x_n \to a\ (n \to +\infty)$ である $\{x_n\}$ に対して

$$f(x_n) \longrightarrow A \quad (n \to +\infty)$$

$$\lim_{x \to a} f(x) = A.$$

極限については後から.



Δy yの増分

$$AP\text{の傾き} = \frac{\boxed{f(x) - f(\alpha)}}{\boxed{x - \alpha}} \longrightarrow f'(\alpha) \quad (x \to \alpha)$$

Δx xの増分

$x = \alpha$ におけるfの微分係数.

‖

$x = \alpha$ における接線の傾き.

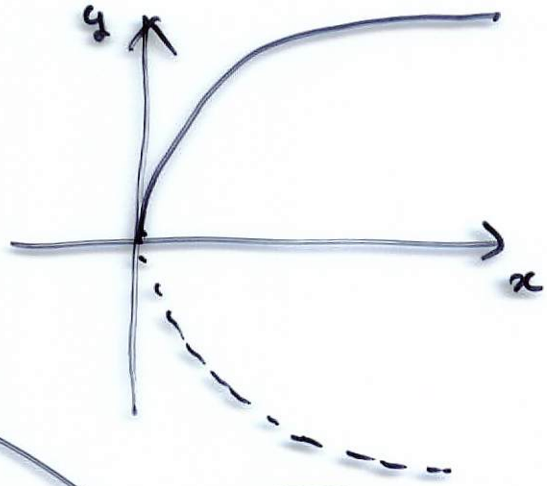
$$= \frac{\Delta y}{\Delta x}$$

AP間の平均変化率

微分商.

$y = \sqrt{x}.\quad (x \geqq 0)$

$y^2 = x$



$a > 0$

$$\lim_{x \to a} \sqrt{x} = \sqrt{a}.$$

近く示す.

$a > 0$

$$\frac{\sqrt{x} - \sqrt{a}}{x - a} = \frac{\sqrt{x} - \sqrt{a}}{x - a} \times \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}$$

$$= \underbrace{\frac{x - a}{x - a}}_{1} \times \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{\sqrt{x} + \sqrt{a}} \to \frac{1}{2\sqrt{a}}$$

$$x_n \to a \Rightarrow \sqrt{x_n} \to \sqrt{a}.$$

$$\sqrt{x_n} + \sqrt{a} \to \sqrt{a} + \sqrt{a} = 2\sqrt{a}$$

$$\frac{1}{\sqrt{x_n} + \sqrt{a}} \to \frac{1}{2\sqrt{a}}$$

$$(\sqrt{x})' = \frac{1}{2} \frac{1}{\sqrt{x}}.$$

$y = x^n$ $\quad \boxed{x = \alpha + h.} \quad x \to \alpha \iff h \to 0$

$$\frac{x^n - \alpha^n}{x - \alpha} = \frac{(\alpha + h)^n - \alpha^n}{h}$$

$$= \frac{\alpha^n + {}_nC_1 \alpha^{n-1} h + \cdots\cdots + h^n - \alpha^n}{h}$$

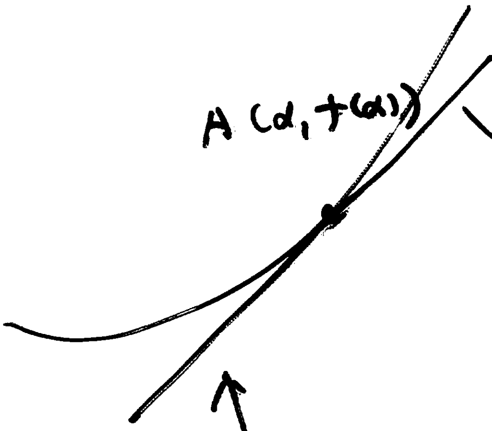
$$= \underbrace{{}_nC_1 \alpha^{n-1}}_{= n\alpha^{n-1}} + \underbrace{{}_nC_2 \alpha^{n-2} h}_{\to 0} + \underbrace{\cdots + h^n}_{\to 0}$$

$$\longrightarrow n \alpha^{n-1}$$

$$\boxed{(x^n)' = n x^{n-1}}$$

明日 $\left\{ \dfrac{x}{(1+x)} \right\}' = ?$, $\left\{ (1+x^2)^{100} \right\}' = ?$

e ネピアの数.

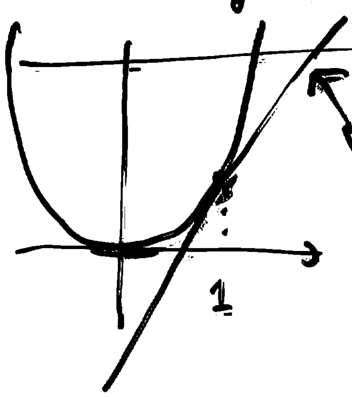


$y = f'(\alpha)(x-\alpha) + f(\alpha)$

接線の傾き $= f'(\alpha)$

1次近似.　　10年複利.　　0.01.

$$M \longmapsto M(1+0.01)^{10} \fallingdotseq M(1+10\times 0.01) = M(1.1)$$



$(x^{10})' = 10x^9$

$y = 10(x-1)+1$. 接線の方程式.

$$(1.01)^{10} \fallingdotseq 10\times(1.01-1)+1 = 1.1$$ 1次近似.

$x \sim \alpha$

$$f(x) \fallingdotseq f(\alpha) + f'(\alpha)(x-\alpha)$$

1次近似.

$\sqrt{1+h} \fallingdotseq 1 + \frac{1}{2}h.$ $\sqrt{1.1} \fallingdotseq 1.05.$

$y = \sqrt{x}.$ $(\sqrt{x})' = \frac{1}{2}\frac{1}{\sqrt{x}}.$

$x = 1+h$

$\alpha = 1$

$$\boxed{f(x) \fallingdotseq f(\alpha) + f'(\alpha)(x-\alpha)}$$

$$\sqrt{1+h} \fallingdotseq 1 + \frac{1}{2}h.$$

⇝ Taylor展開のまねでやる.

$\sqrt{5} \fallingdotseq 2.2360679\cdots\cdots$

$2^2 = 4$ $3^2 = 9$

$2.$

$\sqrt{2} = 1.41421356\cdots\cdots$

$1^2 = 1$ $2^2 = 4$

$(1.4)^2 = 1.96$ $(1.5)^2 = 2.25$

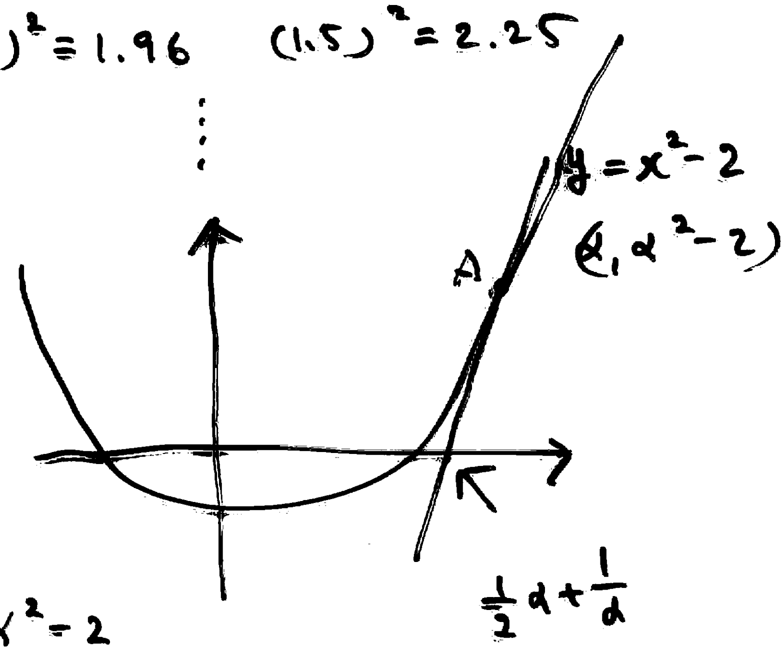
$\vdots$

Newton法

$(x^2-2)' = 2x$

接線の方程式

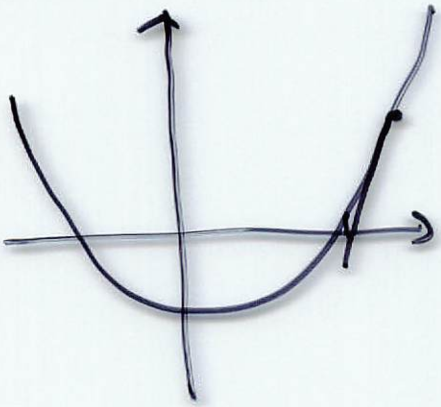
$y = 2\alpha(x-\alpha) + \alpha^2 - 2$



$$2\alpha(x-\alpha)+\alpha^2-2=0$$

$$x-\alpha=-\frac{1}{2\alpha}(\alpha^2-2)$$

$$=-\frac{1}{2}\cdot\alpha+\frac{1}{\alpha} \rightsquigarrow x=\frac{1}{2}\alpha+\frac{1}{\alpha}$$



$y=x^2-2$

$x_1=2.$

$$x_2=\frac{1}{2}x_1+\frac{1}{x_1}$$

$$=1.5.=\frac{3}{2}$$

$$x_3=\frac{1}{2}x_2+\frac{1}{x_2}$$

$$=0.75+\frac{2}{3}=\frac{3}{4}+\frac{2}{3}$$

$$=0.75+0.6666\cdots$$

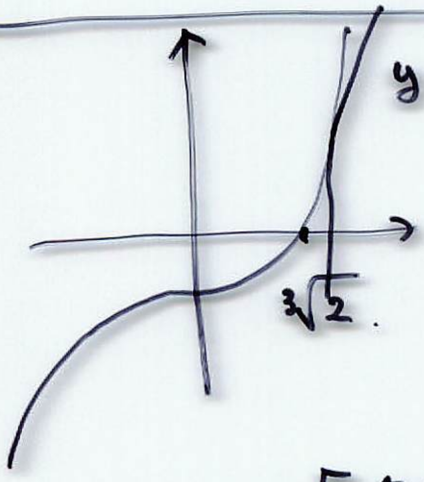
$$=1.41666\cdots$$

$$x_{n+1}=\frac{1}{2}x_n+\frac{1}{x_n}$$

漸化式.

<u>宿題</u>　x_4, x_5 を計算する.

分数, 小数



$y=x^3-2.$

<u>レポート</u>　$\sqrt[3]{2}$ の近似値を Newton 法で求める.

$$\boxed{(x^3-2)'=3x^2.}$$

「分数ができない大学生」

極限

$$\lim_{x \to a} f(x) = A.$$

$\iff$ $x_n \to a \ (n \to +\infty)$ をみたす $\{x_n\}$ に対して

$$f(x_n) \longrightarrow A \quad (n \to +\infty)$$

定理 $\displaystyle\lim_{x \to a} f(x) = A, \ \lim_{x \to a} g(x) = B$

$x \to a$ のとき.

1) $f(x) \pm g(x) \longrightarrow A \pm B$

2) $f(x)\, g(x) \longrightarrow A \cdot B.$

3) $f(x) \neq 0$ 全ての x, $A \neq 0$ とする.

$$\frac{1}{f(x)} \longrightarrow \frac{1}{A}.$$

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2) だけ示す. $x_n \longrightarrow a.$

$$f(x_n)\, g(x_n) \longrightarrow A \cdot B$$

定理 (はさみうち) $f(x) \leq h(x) \leq g(x)$ 全ての x.

$x \to a$ のとき $f(x) \to A, \ g(x) \to A$

$$\Longrightarrow h(x) \longrightarrow A.$$

はさみうち

$x_n \to a$

$$\begin{array}{ccccc} f(x_n) & \leq & h(x_n) & \leq & g(x_n) \\ \downarrow & & \downarrow & & \downarrow \\ A & & A & & A \end{array}$$

定理を使って

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

etc. を示す.

レポートの解説

$$\sum_{k=1}^{n} k^3 = ?$$

$$k^3 - k = k(k^2-1) = (k+1)k(k-1)$$

$$= \frac{1}{4}\left\{(k+2)\underbrace{(k+1)k(k-1)} - \underbrace{(k+1)k(k-1)}(k-2)\right\}$$

$$\begin{array}{lll}
3\cdot\cancel{2\cdot 1}\cdot 0 & - \boxed{2\cdot 1\cdot 0\cdot(-1)} & k=1 \\
4\cdot\cancel{3\cdot 2}\cdot 1 & - 3\cdot\cancel{2\cdot 1}\cdot 0 & k=2 \\
5\cdot\cancel{4\cdot 3}\cdot 2 & - 4\cdot\cancel{3\cdot 2}\cdot 1 & \\
\vdots & \quad\vdots & \\
(n+2)(n+1)n(n-1) & & k=n \\
 & -(n+1)\cancel{n(n-1)}(n-2) & \\
\hline
(n+2)(n+1)n(n-1) & &
\end{array}$$

ベルヌーイの多項式.

$$\sum_{k=1}^{n}(k^3-k) = \frac{(n+2)(n+1)n(n-1)}{4}$$

$$\sum_{k=1}^{n} k^3 = \frac{(n+2)(n+1)n(n-1)}{4} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{4}\left\{(n+2)(n-1)+2\right\}$$

$$= \frac{n(n+1)}{4}(n^2+n) = \frac{n^2(n+1)^2}{4}.$$

Exo

1) $y = \dfrac{1}{1+x}$ の $x = \alpha \ (\neq -1)$ における微分係数

2) $y = \dfrac{1}{(x-1)^2}$ の $x = \alpha \ (\neq 1)$ における微分係数.

3) $y = \dfrac{1}{1+x^2}$ の $x = \alpha$ における微分係数.
