行列の転置.

$$A = \begin{pmatrix} a & \alpha \\ b & \beta \\ c & \gamma \end{pmatrix} \rightsquigarrow {}^tA = \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \rightsquigarrow {}^t({}^tA) = \begin{pmatrix} a & \alpha \\ b & \beta \\ c & \gamma \end{pmatrix} = A$$

3列 2行 2列

3行2列
3×2

transposition 転置.

2行3列.

内積 $\vec{x}, \vec{y} \in \mathbb{R}^n$ $\qquad \displaystyle\sum_{i=1}^{n} x_i y_i$

$$(\vec{x}, \vec{y}) = {}^t\vec{x}\,\vec{y}$$

$$= (x_1 \cdots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$1 \times n$ $\quad$ $n \times 1$ $\quad$ 1×1

転置の性質.

$$= (y_1 \cdots y_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

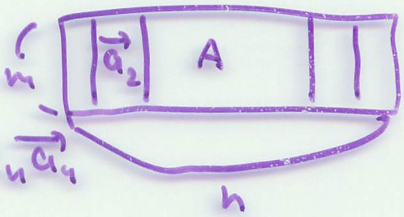
1° ${}^t({}^tA) = A$

2° $A : m \times n$ $\quad$ $B : n$行 l列 $\rightsquigarrow AB \quad m \times l$

m行n列 $\qquad n \times l$

$$AB = A(\vec{b}_1 \cdots \vec{b}_l) = \underbrace{(A\vec{b}_1 \cdots A\vec{b}_l)}_{l \text{列}} \Big\} m$$

$A\vec{b}_j \in \mathbb{R}^m$



$$A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

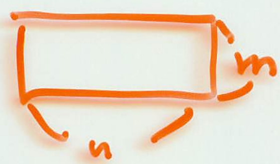
$\vec{x} \in \mathbb{R}^n$

2°

$A : m \times \underset{\sim}{n}$ $\quad B : \underset{\substack{\sim \\ 行}}{n} \times \underset{列}{l}$ $\rightsquigarrow$ $AB \ \ m \times l$

$$AB = A\underbrace{(\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_l)}_{l\ 列}\Big\}\, n$$

$$= \underbrace{(A\vec{b}_1\ A\vec{b}_2 \cdots A\vec{b}_l)}_{l\ 列}\Big\}\, m$$

$A =$ 

$$A\vec{x} = (\vec{a}_1 \cdots \vec{a}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$x \in \mathbb{R}^n$

$$= x_1\vec{a}_1 + \cdots + x_n\vec{a}_n \in \mathbb{R}^m$$

${}^t(AB) \quad l \times m$ $\qquad A : m \times \underset{\sim}{n}\ \ B : \underset{\sim}{n} \times l$

$$\left.\begin{array}{l} {}^tB : l \times n \\ {}^tA : n \times m \end{array}\right\} \longrightarrow {}^tB\,{}^tA \quad l \times m$$

<u>公式</u>

$$ {}^t(AB) = {}^tB\,{}^tA. $$

$AB \quad m \times l \rightsquigarrow {}^t(AB) \quad l \times m$

$A : m \times n \rightsquigarrow {}^tA : n \times m$

$B : n \times l \rightsquigarrow {}^tB : l \times m$

$\rightarrow {}^tB\,{}^tA$

$l \times m$

<u>公式</u> $\quad {}^t(AB) = {}^tB\,{}^tA$

$$A = \begin{pmatrix} \boxed{\mathbf{a}_1} \\ \vdots \\ \boxed{\mathbf{a}_m} \end{pmatrix}$$

$m \times n$

$$B = \overbrace{\begin{pmatrix} \vec{b}_1 & \cdots & \vec{b}_l \end{pmatrix}}^{l\text{列}}$$

$n \times l$

$$ {}^tB\,{}^tA = \begin{array}{c} \begin{pmatrix} \boxed{{}^t\vec{b}_1} \\ {}^t\vec{b}_2 \\ \vdots \\ {}^t\vec{b}_l \end{pmatrix} j \end{array} \begin{array}{c} \begin{pmatrix} {}^t\mathbf{a}_1 & {}^t\mathbf{a}_2 & \cdots & {}^t\mathbf{a}_m \end{pmatrix} \\ i \end{array} $$

$\rightarrow$ j行i列 $= {}^t\vec{b}_j\,{}^t\mathbf{a}_i = \mathbf{a}_i \vec{b}_j$

$$AB = \begin{pmatrix} \vdots \\ \boxed{\mathbf{a}_i} \\ \vdots \end{pmatrix} \begin{pmatrix} \cdots & \boxed{\vec{b}_j} & \cdots \end{pmatrix}$$

$\rightarrow$ i行j列 $= \boxed{\mathbf{a}_i}\,\boxed{\vec{b}_j}$

---

t

${}^t(AB)$: j行i列 $= (AB)$のi行j列

$= \mathbf{a}_i \vec{b}_j$

何故 ${}^tA$ を使うか? 内積のため!!

$$A : \overset{行}{m} \times \overset{列}{n}. \quad \vec{x} \in \mathbb{R}^n$$

$$= \left.(\underbrace{\vec{a}_1 \cdots \vec{a}_n}_{n個})\right\}m \qquad \vec{a}_j \in \mathbb{R}^m$$

$$A\vec{x} = (\vec{a}_1 \cdots \vec{a}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{x_1\vec{a}_1 + \cdots + x_n\vec{a}_n}_{\mathbb{R}^m}$$

公式 $\vec{y} \in \mathbb{R}^m$

$$(\underbrace{A\vec{x}, \vec{y}}_{内積}) = (\vec{x}, {}^tA\vec{y})$$

$${}^tA : n \times m, \ \vec{y} \in \mathbb{R}^m \rightsquigarrow {}^tA\vec{y} \in \mathbb{R}^n$$

証明

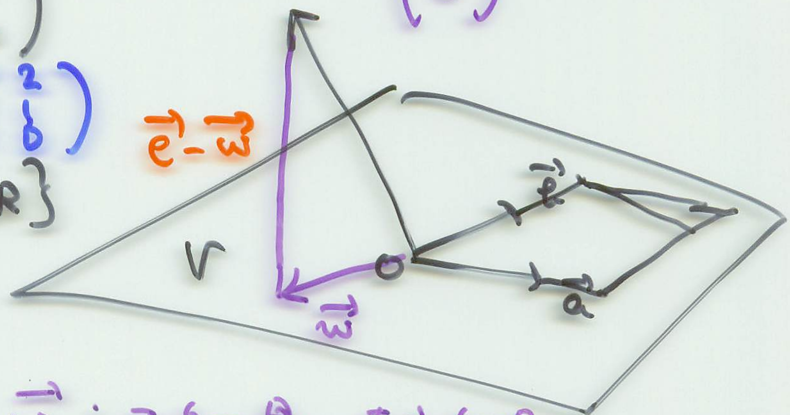
$$(左) = {}^t(A\vec{x})\vec{y} = {}^t\vec{x}({}^tA\vec{y})$$

$$= (\vec{x}, {}^tA\vec{y}) = (右).$$

$\vec{a} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $\quad$ $\vec{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$A = (\vec{a}\ \vec{b}) = \begin{pmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$

$V = \{ s\vec{a} + t\vec{b} \,;\, s, t \in \mathbb{R} \}$



$\vec{e} - \vec{w} \perp V$

$\vec{w} \in V$

$(\vec{e} - \vec{w}, \vec{v}) = 0$ $\quad$ $\vec{w}$: 正射影, 直交射影

(全ての $\vec{v} \in V$)

$\vec{w} = s\vec{a} + t\vec{b}$, $\quad$ $\vec{v} \in V$ $\quad$ $(\vec{a}\ \vec{b})\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$= (\vec{a}\ \vec{b})\begin{pmatrix} s \\ t \end{pmatrix}$ $\quad$ $\vec{v} = \alpha\vec{a} + \vec{b}\beta$ と書ける

$(\vec{e} - \vec{w}, \vec{v}) = 0$ 全ての α, β で成立

$= \left(\vec{e} - A\begin{pmatrix} s \\ t \end{pmatrix}, A\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$

$= \left({}^tA\vec{e} - {}^tAA\begin{pmatrix} s \\ t \end{pmatrix} ; \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$

任意の $\mathbb{R}^2$ のベクトル

$\rightsquigarrow$ ${}^tA\vec{e} - {}^tAA\begin{pmatrix} s \\ t \end{pmatrix} = \vec{0}$

$$\boxed{(\vec{x}, \vec{y}) = 0 \ (\text{全ての } \vec{y} \in \mathbb{R}^n) \iff \vec{x} = \vec{0}}$$

${}^tAA\begin{pmatrix} s \\ t \end{pmatrix} = {}^tA\vec{e}$ $\quad$ 正規方程式

${}^tAA = \begin{pmatrix} 2 & 1 \\ 5 & 1 \end{pmatrix}$ $\quad$ $\det({}^tAA) = 2\times 1 - 5\times 1$

$= -3$

より

$$\boxed{\begin{pmatrix} s \\ t \end{pmatrix} = ({}^tAA)^{-1}\, {}^tA\vec{e}}$$

$${}^tAA\begin{pmatrix}s\\t\end{pmatrix} = {}^tA\vec{e}$$

$${}^tAA = \begin{pmatrix}{}^t\vec{a}\\ {}^t\vec{b}\end{pmatrix}\begin{pmatrix}\vec{a} & \vec{b}\end{pmatrix} = \begin{pmatrix}{}^t\vec{a}\vec{a} & {}^t\vec{a}\vec{b}\\ {}^t\vec{b}\vec{a} & {}^t\vec{b}\vec{b}\end{pmatrix} = \begin{pmatrix}\|\vec{a}\|^2 & (\vec{a},\vec{b})\\ (\vec{b},\vec{a}) & \|\vec{b}\|^2\end{pmatrix}$$

(Aが定めるグラム行列)

$$= \begin{pmatrix}0 & 1 & 1\\ 2 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 2\\ 1 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 5\end{pmatrix}$$

$\det({}^tAA) = 2\times 5 - 1 = 9 \neq 0$ より

$$\begin{pmatrix}s\\t\end{pmatrix} = ({}^tAA)^{-1}\,{}^tA\vec{e}$$

$$= \frac{1}{9}\begin{pmatrix}5 & -1\\ -1 & 2\end{pmatrix}\cdot\begin{pmatrix}0 & 1 & 1\\ 2 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\boxed{\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1} = \frac{1}{D}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix} \qquad D = ad - bc \neq 0}$$

$= \cdots\cdots$

$${}^tAA = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \cdots$$

I $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ とする.

$V = \{ s\vec{a} + t\vec{b} \ ; \ s, t \in \mathbb{R} \}$ とおく.

$\vec{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ の V への正射影を

$\vec{w} = s\vec{a} + t\vec{b}$ の形で求めよ.

II
$$\begin{cases} x + y + 3z + 2u + v = 0 \\ 5x + y + 8z + 6u + 3v = 0 \\ 3x - y + 2z + 2u + v = 0 \end{cases}$$

を解け.

$$\left(\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 1 & 0 \\ 5 & 1 & 8 & 6 & 3 & 0 \\ 3 & -1 & 2 & 2 & 1 & 0 \end{array}\right)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$ad - bc \neq 0$