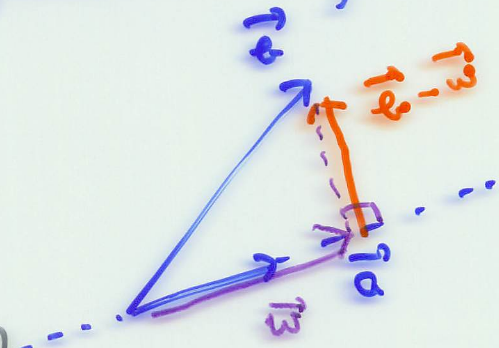
11/10 線形代数 2004年

$\vec{b}$ の $\vec{a}$ 方向への正射影 $\vec{w}$

(normal projection)

(orthogonal ———)



$$\left.\begin{aligned} &\vec{w} = t\vec{a} \\ &\vec{b}-\vec{w} \perp \vec{a} \end{aligned}\right\}$$

$(\vec{b}-t\vec{a}, \vec{a}) = 0$

$$\boxed{\vec{w} = \frac{(\vec{b}, \vec{a})}{\|\vec{a}\|^2}\vec{a}.}$$

$$\vec{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

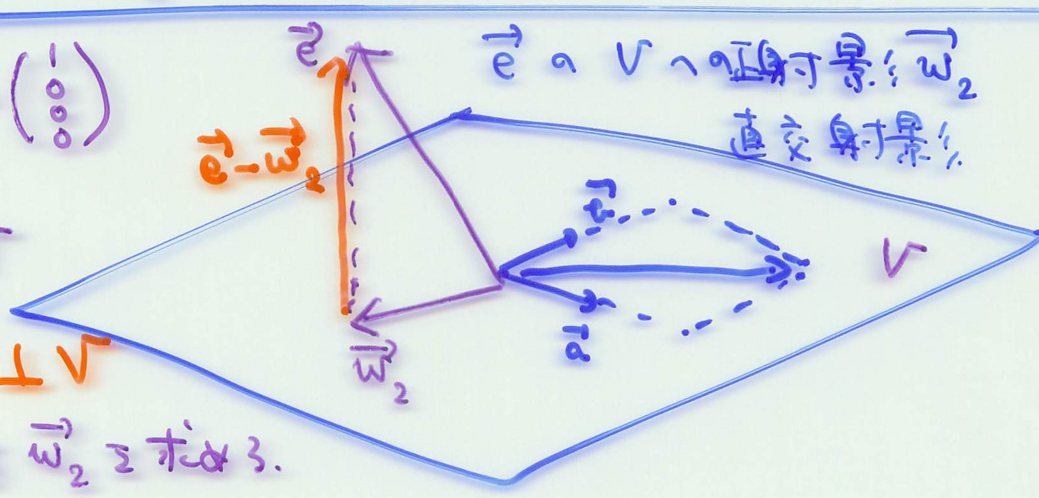
$$(\vec{a}, \vec{b}) = 3\times 1 + 2\times 1 + 1\times(-1) + (-2)\times(-1)$$
$$= 6$$

$$\vec{w} = \frac{6}{4}\vec{a} = \frac{3}{2}\vec{a}$$

$$\vec{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\vec{e}$ の V への正射影 $\vec{w}_2$

直交射影



$$\left(\begin{aligned} &\vec{w}_2 \in V \\ &\vec{e}-\vec{w}_2 \perp V \end{aligned}\right.$$

をみたす $\vec{w}_2$ を求める.

$$V = \{ s\vec{a} + t\vec{b} \,;\, s, t \in \mathbb{R} \}$$

$\vec{a}$ と $\vec{b}$ とが張る平面

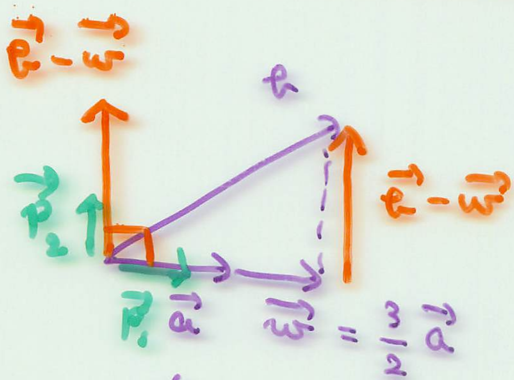
$$\vec{w}_2 = s\vec{a} + t\vec{b}.$$

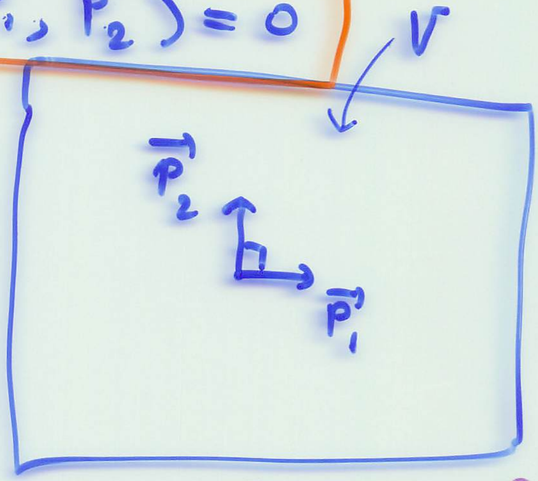
$$(\vec{w}_2 - \vec{e}, \vec{a}) = (\vec{w}_2 - \vec{e}, \vec{b}) = 0$$

$\longrightarrow$ s と t を求める.

$V \to \vec{p}_1, \vec{p}_2$

$$\boxed{\|\vec{p}_1\| = \|\vec{p}_2\| = 1, \quad (\vec{p}_1, \vec{p}_2) = 0}$$





$$\vec{w} = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{b} - \vec{w} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ \frac{5}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 3 \\ 1 \\ 5 \\ -1 \end{pmatrix}$$

$$\vec{p}_1 = \frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

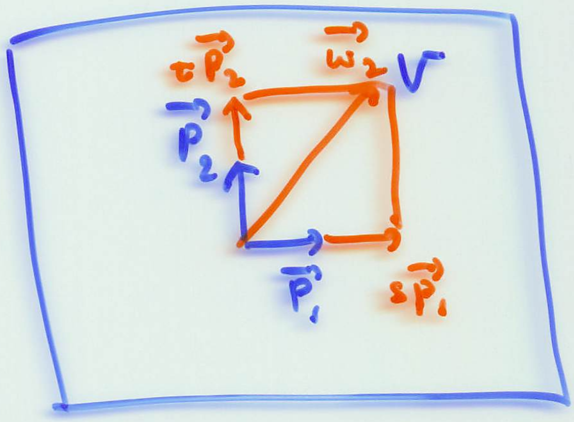
$$\vec{p}_2 = \frac{1}{6}\begin{pmatrix} 3 \\ 1 \\ 5 \\ -1 \end{pmatrix}$$

V の正規直交基底

$$= \frac{1}{\|\vec{b} - \vec{w}\|}(\vec{b} - \vec{w})$$

$$V = \{ s\vec{p}_1 + t\vec{p}_2 \ ; \ s, t \in \mathbb{R} \}$$

とおいている。 $\vec{w}_2 \in V$ なので



$$\vec{w}_2 = s\vec{p}_1 + t\vec{p}_2$$

とおける.

$$(\vec{e} - \vec{w}_2) \perp V$$

$$\Updownarrow$$

$$(\vec{e} - s\vec{p}_1 - t\vec{p}_2, \vec{p}_1) = (\vec{e} - s\vec{p}_1 - t\vec{p}_2, \vec{p}_2) = 0$$

$$\| $$

$$(\vec{e}, \vec{p}_1) - s\underbrace{\boxed{(\vec{p}_1, \vec{p}_1)}}_{=1} - t\underbrace{\boxed{(\vec{p}_1, \vec{p}_2)}}_{=0}$$

$$(\vec{e}, \vec{p}_2) - s\overbrace{\boxed{(\vec{p}_1, \vec{p}_2)}}^{0 =} - t\underbrace{\boxed{(\vec{p}_2, \vec{p}_2)}}_{=1} = 0$$

$$\rightsquigarrow \quad s = (\vec{e}, \vec{p}_1)$$

$$t = (\vec{e}, \vec{p}_2)$$

$$\boxed{\vec{w}_2 = \overset{s}{\overset{\|}{(\vec{e}, \vec{p}_1)}}\vec{p}_1 + \overset{t}{\overset{\|}{(\vec{e}, \vec{p}_2)}}\vec{p}_2}$$

コーシー・シュワルツの不等式

$\vec{a}, \vec{b} \in \mathbb{R}^n$

相関係数

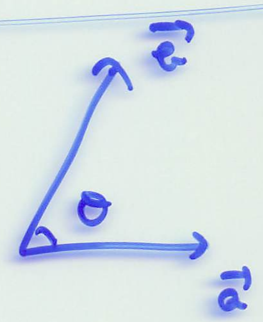
$$|(\vec{a}, \vec{b})| \leqq \|\vec{a}\| \cdot \|\vec{b}\|$$

$$\Updownarrow$$

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leqq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$$

$n = 2, 3$ のときは easy, facile

$$(\vec{a}, \vec{b}) = \|\vec{a}\| \|\vec{b}\| \cos\theta$$



$$|(\vec{a}, \vec{b})| = \|\vec{a}\| \|\vec{b}\| |\cos\theta| \leqq \|\vec{a}\| \|\vec{b}\|$$

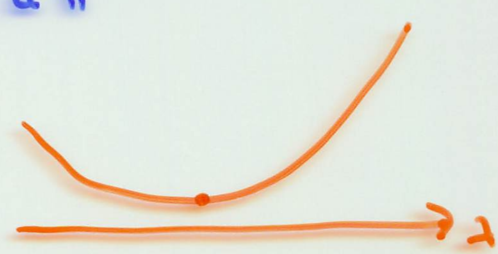
discriminant

$$\|\vec{a} + \lambda \vec{b}\|^2 \geqq 0 \leftarrow \lambda \text{ に依らず}$$

$$= \lambda^2 \overset{1°}{\|\vec{b}\|^2} + 2\lambda(\vec{a}, \vec{b}) + \|\vec{a}\|^2$$

$\vec{b} = \vec{0}$ のとき 不等式は自明

$\vec{b} \neq \vec{0}$ のとき $\|\vec{b}\|^2 > 0$



$\frac{D}{4}$

$$D/4 = (\vec{a}, \vec{b})^2 - \|\vec{a}\|^2 \|\vec{b}\|^2 \leqq 0$$

$$|(\vec{a}, \vec{b})| \leqq \|\vec{a}\| \|\vec{b}\|.$$

$$\|\vec{a} \pm \vec{b}\|^2 = \|\vec{a}\|^2 \pm 2(\vec{a}, \vec{b}) + \|\vec{b}\|^2$$

$\|\vec{a}+\lambda\vec{b}\|^2 \geqq 0$ ← 全ての $\lambda \in \mathbb{R}$ で成立

$= \|\lambda\vec{b}\|^2 + 2(\vec{a}, \lambda\vec{b}) + \|\vec{a}\|^2$

$= \lambda^2\|\vec{b}\|^2 + 2\lambda(\vec{a}, \vec{b}) + \|\vec{a}\|^2$

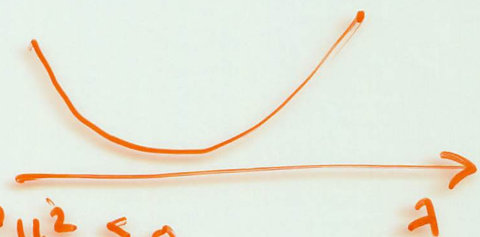
λ の2次式.

$\vec{b} = \vec{0}$ のとき 不等式は自明.

$$\boxed{\|\vec{a} \pm \vec{b}\|^2 = \|\vec{a}\|^2 \pm 2(\vec{a}, \vec{b}) + \|\vec{b}\|^2}$$

$\|\lambda\vec{b}\| = |\lambda|\,\|\vec{b}\|$

$\vec{b} \neq \vec{0}$ のとき $\|\vec{b}\|^2 > 0$



$$\frac{D}{4} = (\vec{a}, \vec{b})^2 - \|\vec{b}\|^2\|\vec{a}\|^2 \leqq 0$$

D: 判別式. discriminant

$$(\vec{a}, \vec{b})^2 \leqq \|\vec{a}\|^2\|\vec{b}\|^2$$

① 1次方程式, 行列の基本変形, 逆行列の求め方

② 行列式 (3次, 4次) とその応用.

拡大行列

$$\begin{cases} x+2y-z=3 & \cdots ① \\ x+4y-5z=1 & \cdots ② \\ 3x+7y-5z=8 & \cdots ③ \end{cases} \longleftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & 4 & -5 & 1 \\ 3 & 7 & -5 & 8 \end{array}\right)$$

$\updownarrow$

$$\begin{cases} x+2y-z=3 & \cdots ①' \\ 2y-4z=-2 & \cdots ②' = ② + ① \times (-1) \\ \cancel{-3y+12z=5} & \cdots ③' = ③ + ① \times (-3) \\ y-2z=-1 & \end{cases}$$

2r += 1r × (−1)

3r += 1r × (−3)

$$\cancel{\begin{array}{r} 3x+7y-5z=8 \\ -)\ 3x+12y-15z=3 \\ \hline -5y+10z=5 \end{array}}$$

$$\begin{array}{r} 3x+7y-5z=8 \\ -)\ 3x+6y-3z=9 \\ \hline y-2z=-1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 2 & -4 & -2 \\ 0 & 1 & -2 & -1 \end{array}\right)$$

2r × $\frac{1}{2}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -2 & -1 \end{array}\right)$$

3r += 2r × (−1)

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$0x+0y+0z=0$

$$\begin{array}{rrrr} 1 & 2 & -1 & 3 \\ -)\ 0 & 2 & -4 & -2 \\ \hline 1 & 0 & 3 & 5 \end{array}$$

1r += 2r × −2.

pivot

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -1 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \end{array}\right)$$

階段行列

$z=\alpha$ とおく.

$$\begin{cases} x+3z=5 \\ y-2z=-1 \end{cases}$$

$$\begin{cases} x=-3\alpha+5 \\ y=2\alpha-1 \\ z=\alpha \end{cases}$$

と解ける

Gaussの掃き出し法

連立1次方程式　掃き出し法 (Gauss)

$$\begin{cases} x + 2y - z = 3 & \cdots ① \\ x + 4y - 5z = 1 & \cdots ② \\ 3x + 7y - 5z = 8 & \cdots ③ \end{cases} \longleftrightarrow$$

拡大行列

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & 4 & -5 & 1 \\ 3 & 7 & -5 & 8 \end{array}\right)$$

r 行
row

$$\updownarrow$$

2r += 1r × (−1)
3r += 1r × (−3)

$$\begin{cases} x + 2y - z = 3 & \cdots ①' = ① \\ \quad 2y - 4z = -2 & \cdots ②' = ② + ① \times (-1) \\ \quad y - 2z = -1 & ③' = ③ + ① \times (-3) \end{cases}$$

$$\begin{array}{r} 3x + 7y - 5z = 8 \\ -)\ 3x + 6y - 3z = 9 \\ \hline y - 2z = -1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 2 & -4 & -2 \\ 0 & 1 & -2 & -1 \end{array}\right)$$

2r × = $\frac{1}{2}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -2 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

3r += 2r × (−1)

$$\boxed{\begin{array}{l} x += y \\ x \text{ を } x+y \text{ に} \\ \quad\text{する} \end{array}}$$

1r += 2r × (−2)

$$\begin{array}{r} 1\ \ 2\ -1\ \ 3 \\ -)\ 0\ \ 2\ -4\ -2 \\ \hline 1\ \ 0\ \ \ 3\ \ \ 5 \end{array}$$

$$\begin{array}{c} \left(\begin{array}{ccc|c} \cancel{1} & \cancel{2} & \cancel{-1} & \cancel{3} \\ 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

○
○
pivot

$$0x + 0y + 0z = 0$$

$$\begin{cases} x \qquad + 3z = 5 \\ \qquad y - 2z = -1 \end{cases}$$

$z = \alpha$ とおく

$$\begin{cases} x = 5 - 3\alpha \\ y = -1 + 2\alpha \\ z = \alpha \end{cases}$$

と解ける.

$$
\begin{array}{c}
\begin{array}{ccc} x & y & z \end{array} \\
\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & 4 & -5 & 2 \\ 3 & 7 & -5 & 4 \end{array}\right)
\end{array}
\xrightarrow[3r \mathrel{+}= 1r \times (-3)]{2r \mathrel{+}= 1r \times (-1)}
\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 2 & -4 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right)
$$

$$
\xrightarrow{2r \times= \frac{1}{2}}
\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & \frac{1}{2} \\ 0 & 1 & -2 & 1 \end{array}\right)
\longrightarrow
\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{array}\right)
$$

行基本変形.

Ⅰ $i \neq j$ i行のλ倍をj行に+

Ⅱ $i \neq j$ i行とj行を交換.

Ⅲ $\lambda \neq 0$ i行をλ倍

$0x + 0y + 0z = \frac{1}{2}$ → 解なし

解けないものもある.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & 4 & -5 & 2 \\ 3 & 7 & -5 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 2 & -4 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

2r + = 1r × (−1)
3r + = 1r × (−3)

$\downarrow$ 2r × = $\frac{1}{2}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & \frac{1}{2} \\ 0 & 1 & -2 & 1 \end{array}\right)$$

$\downarrow$ 3r + = 2r × (−1)

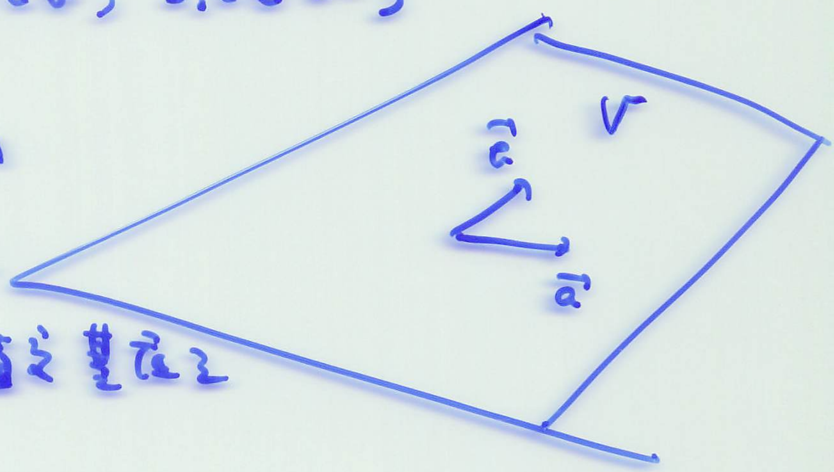
$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$0x + 0y + 0z = \frac{1}{2}$ → 解なし.

I $\vec{a} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ が張る平面を V とする

$$V = \{ s\vec{a} + t\vec{b} \,;\, s, t \in \mathbb{R} \}$$

$\vec{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ の V への正射影を求める.



まず V の正規直交基底を求める

$V \ni \vec{p_1}, \vec{p_2}$ で $\left.\begin{array}{l} \|\vec{p_1}\| = \|\vec{p_2}\| = 1 \\ \langle \vec{p_1}, \vec{p_2} \rangle = 0 \end{array}\right\}$ をみたすもの.