$$\left(\begin{array}{ccc|c} ① & 1 & a & 1 \\ 1 & a & 1 & 3 \\ a & 1 & 1 & 2a \\ x & y & z & \end{array}\right) \xrightarrow[3r += 1r \times (-a)]{2r += 1r \times (-1)} \left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & \boxed{a-1} & 1-a & 2 \\ 0 & 1-a & 1-a^2 & a \end{array}\right)$$

$\downarrow$ 3r += 2r × 1

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 2 \\ 0 & 0 & & a+2 \end{array}\right)$$

$$1-a^2+(1-a)$$
$$= 2-a-a^2$$
$$= -(a^2+a-2)$$
$$= -(a+2)(a-1)$$

$a=1$ のとき.

$$1 = \mathrm{rank}(A) < \mathrm{rank}(A|\vec{b}) = 2$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & & 2 \\ 0 & 0 & 0 & & -1 \end{array}\right)$$

$0x+0y+0z=2$

解なし.

$a \neq 1$

2r × = $\frac{1}{a-1}$

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & 1 & -1 & \frac{2}{a-1} \\ 0 & 0 & & a+2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & 1 & -1 & \frac{2}{a-1} \\ 0 & 0 & 1 & \frac{-1}{a-1} \end{array}\right)$$

$-(a+2)(a-1)$

$a \neq -2$

$\mathrm{rank}(A) = \mathrm{rank}(A|\vec{b})$

⋮

解あり

$a=-2$ のとき

$$= \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & ① & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\mathrm{rank}(A)$
$= \mathrm{rank}(A|\vec{b})$

$\downarrow$ 1r += 2r × (−1)

$z = \alpha$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

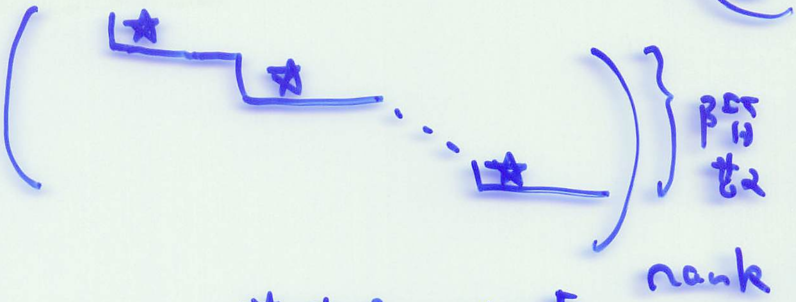
$$\begin{cases} x = -3\alpha - 1 \\ y = \alpha + 2 \\ z = \alpha \end{cases}$$

パラメータ. 1個

まとめ

$A \xrightarrow{} \cdots \xrightarrow{} A_0 =$

行基本変形



★ ≠ 0 pivot

$A\vec{x} = \vec{b}$ に解あり $\Longleftrightarrow$ $\operatorname{rank}(A) = \operatorname{rank}(\vec{b})$

$A : m \times \underset{\sim}{n}$ 行列

解があるとき [illegible] 変数 n 個

($n - \operatorname{rank}(A)$ 個のパラメーターで
解が書ける.

$\operatorname{Ker}(A)$ の次元

12.5

逆行列

A : n次正方行列.
$n \times n$

$AX = XA = I_n$ をみたす n次正方行列 があるとき A : 正則

① 逆行はあると唯一つ.

$$\left.\begin{array}{l} AX = XA = I_n \\ AY = YA = I_n \end{array}\right\} \rightsquigarrow X = Y$$

$$AX = I_n \rightsquigarrow Y(AX) = YI_n = Y$$

$$\overset{\shortparallel}{} \leftarrow \text{結合則}$$

$$(YA)X = I_n X = X$$

---

② O_3 は正則でない.

$$O_3 X = O_3 , \quad XO_3 = O_3$$

$$A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \text{ 正則でない}$$

$$A\vec{e}_3 = \begin{pmatrix} 1 & & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}.$$

A : $m \times n$.

$$\vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j$$

$$A\vec{e}_j = \vec{a}_j \quad \text{(j列)}$$

$$A = (\vec{a}_1 \cdots \vec{a}_n)$$

A が正則とすると $\underbrace{A^{-1}A}_{\overset{\shortparallel}{I_3}}\vec{e}_3 = \vec{0} \rightsquigarrow \vec{e}_3 = \vec{0}$ 矛盾.

S_{ij} $\longrightarrow$ $S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ z \end{pmatrix} \leftrightarrows$$

$$S_{12} S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boxed{0\;1\;0} \\ \boxed{1\;0\;0} \\ 0\;0\;1 \end{pmatrix} \circlearrowleft = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$S_{12} S_{12} = I_3$ $\quad$ S_{12} : 正則, $S_{12}^{-1} = S_{12}$

$Q_i(\lambda)$ $\quad$ $Q_2(\lambda) = \begin{pmatrix} 1 & & \\ & \lambda & \\ & & 1 \end{pmatrix}$

$\lambda \neq 0$

$$Q_2(\lambda) Q_2(\mu) = Q_2(\lambda\mu)$$

$$\begin{pmatrix} 1 & & \\ & \lambda & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \mu & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \lambda\mu & \\ & & 1 \end{pmatrix}$$

$$Q_2(\lambda)\, Q_2\left(\frac{1}{\lambda}\right) = Q_2\left(\frac{1}{\lambda}\right) Q_2(\lambda) = Q_2(1)$$
$$= I_3$$

$R_{ij}(\lambda)$

$$R_{12}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow$$

1行のλ倍を2行に加える.

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \mu & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda+\mu & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{12}(\lambda)\,R_{12}(\mu) = R_{12}(\lambda+\mu)$$

$$R_{12}(\lambda)\,R_{12}(-\lambda) = R_{12}(-\lambda)\,R_{12}(\lambda) = I_3$$

$\underline{\lambda \neq 0 \text{ のとき}}$

$R_{12}(\lambda)$ 正則 $\quad R_{12}(\lambda)^{-1} = R_{12}(-\lambda)$

---

1) 基本行列：正則

$(\text{基本行列})^{-1} = \text{基本行列}.$

2) 行基本変形 ＝ 基本行列を左からかける.

II. $$A = \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 3 & 0 & 8 & 0 & 1 & 0 \\ 2 & 3 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\text{行基本変形}]{} \cdots \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} -\frac{24}{10} & \frac{6}{10} & \frac{18}{10} \\ \frac{7}{10} & -\frac{3}{10} & -\frac{9}{10} \\ \frac{9}{10} & -\frac{1}{10} & -\frac{3}{10} \end{matrix}}_{\begin{matrix} \shortparallel \\ A^{-1}. \end{matrix}} \\ 0 & 0 & 1 & \end{array}\right)$$

$$(A \mid I_3) \to \cdots \to (I_3 \mid B)$$

$P =$

$$\boxed{P_l \cdots P_2 P_1}(A \mid I_3) = (I_3 \mid B)$$

P_j : 基本行列
正則.

$\downarrow$

P : 正則

A : 正則

B : 正則 $\Rightarrow$ AB : 正則

A^{-1} : 正則

$$\begin{cases} \underbrace{AB \cdot B^{-1}}A^{-1} = AA^{-1} = I_3 \\ B^{-1}\underbrace{A^{-1}AB} = B^{-1}B = I_3 \end{cases} \rightsquigarrow (AB)^{-1} = B^{-1}A^{-1}$$

$$\to (A^{-1})^{-1} = A$$

inverse.

$$P(A \mid I_3) = (I_3 \mid B)$$

$$\| $$

$$(P\vec{a}_1\ P\vec{a}_2\ P\vec{a}_3 \mid P\vec{e}_1\ P\vec{e}_2\ P\vec{e}_3)$$

$$\| $$

$$(PA \mid P)$$

$\rightsquigarrow$ $PA = I_3,\ P = B$

$\rightsquigarrow$ $BA = I_3$ $\rightsquigarrow$ $AB = I_3$

?

$P^{-1}\cdot$ $P^{-1}PA = P^{-1}I_3$

$\to$ $A = P^{-1}$: 正則 $\quad A^{-1} = (P^{-1})^{-1} = P$

P 正則 $\qquad = B.$

$$A = \underset{\substack{\parallel \\ (\vec{a}_1 \cdots \vec{a}_4)}}{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 6 & 3 & 7 \\ 2 & 1 & 4 & 0 \end{pmatrix}} \xrightarrow{\quad} \cdots \xrightarrow{\quad} \underset{\substack{\parallel \\ B = (\vec{b}_1 \cdots \vec{b}_4)}}{\begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

行基本変形

$$\underset{\substack{\parallel \\ P}}{\boxed{P_\ell \cdots\cdots P_2 P_1}} A = B$$

P 正則.

$$P(\vec{a}_1 \cdots \vec{a}_4) = (\vec{b}_1 \cdots \vec{b}_4)$$

$$\overset{\parallel}{(P\vec{a}_1 \cdots P\vec{a}_4)} \rightsquigarrow P\vec{a}_j = \vec{b}_j$$

$$\vec{a}_j = P^{-1}\vec{b}_j$$

$$\mathrm{Im}(A) = \{ A\vec{x} \in \mathbb{R}^3 ;\ \vec{x} \in \mathbb{R}^4 \}$$

$$\overset{\cap}{\mathbb{R}^3} \quad = \{ x_1\vec{a}_1 + \cdots + x_4\vec{a}_4 ;\ x_1, \cdots, x_4 \in \mathbb{R} \}$$

$$\vec{b}_3 = 3\vec{b}_1 - 2\vec{b}_2, \quad \vec{b}_4 = -\vec{b}_1 + 2\vec{b}_2$$

P^{-1}.

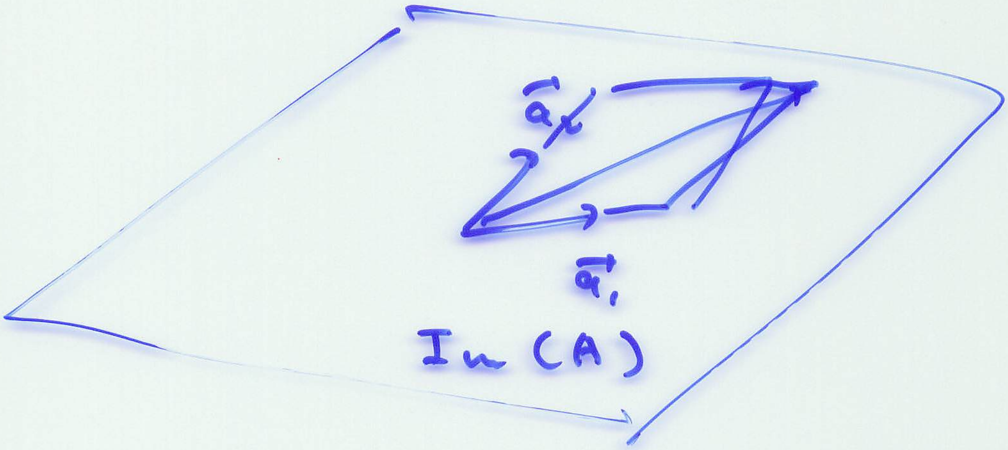
$$P^{-1}\vec{b}_3 = P^{-1}(3\vec{b}_1 - 2\vec{b}_2)$$

$$= 3P^{-1}\vec{b}_1 - 2P^{-1}\vec{b}_2$$

$$\vec{a}_3 = 3\vec{a}_1 - 2\vec{a}_2, \quad \vec{a}_4 = -\vec{a}_1 + 2\vec{a}_2$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\underset{\substack{\parallel \\ 3\vec{a}_1 - 2\vec{a}_2}}{\vec{a}_3} + x_4\underset{\substack{\parallel \\ -\vec{a}_1 + 2\vec{a}_2}}{\vec{a}_4}$$

$$= (x_1 + 3x_3 - x_4)\vec{a}_1 + (x_2 - 2x_3 + 2x_4)\vec{a}_2$$



$$\left(\vec{a}_1 \text{ と } \vec{a}_2 \text{ は平行でない} \right)$$

$$\Longleftrightarrow \left(c_1\vec{a}_1 + c_2\vec{a}_2 = \vec{0} \Longrightarrow c_1 = c_2 = 0 \right)$$

↑ $\vec{a}_1, \vec{a}_2$ は線形独立.

P.

$$c_1 P\vec{a}_1 + c_2 P\vec{a}_2 = \vec{0}$$

$$\rightsquigarrow \quad c_1\vec{e}_1 + c_2\vec{e}_2 = \vec{0} \qquad = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$$

$$\Longrightarrow c_1 = c_2 = 0$$

Im(A) は 2次元

基底として $\vec{a}_1, \vec{a}_2$ がとれる.

(8

I　$A = \begin{pmatrix} 1 & -2 & 5 & 0 \\ -3 & 1 & 2 & -3 \\ 2 & -1 & 1 & 3 \\ 4 & -2 & -3 & 1 \end{pmatrix}$ に対して

$\mathrm{Im}(A)$ の基底を求めよ.

II.　$A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 8 \\ 2 & \underset{\substack{| \\ 3}}{\textcircled{3}} & 3 \end{pmatrix}$ に対して A^{-1} を求めよ.