$z = x^4 + y^4 - 4(x-y)^2$ の 極大値
極小値.

$z_x = 4x^3 - 8(x-y) = 0$ …①

$z_y = 4y^3 + 8(x-y) = 0$ …②

①+② から $x^3 + y^3 = 0$ が必要.

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$
$$= (x+y)\left\{\left(x - \frac{y}{2}\right)^2 + \frac{3}{4}y^2\right\} = 0$$

$\Longleftrightarrow$ $x + y = 0$ OR $\left(x = \frac{y}{2}\ かつ\ y = 0\right)$

$\Longleftrightarrow$ $x = -y$ OR $(0, 0)$

(i) $x = -y$ のとき ①に代入

$4x^3 - 16x = 0 \Longleftrightarrow x = 0, \pm 2$

それぞれの場合 $x = 0, \mp 2$.

(ii) $(0,0)$ は (i)に含まれる.

$(x, y) = (0,0), (2,-2), (-2,2)$

が停留点.

$$z_x = 4x^3 - 8x + 8y$$

$$z_y = 4y^3 + 8x - 8y.$$

$$z_{xx} = \cancel{8x^2}\,12x^2 - 8 = \cancel{8(x^2-1)}\,4(3x^2-2)$$

$$z_{xy} = 8, \quad z_{yx} = 8$$

$$z_{yy} = \cancel{8}\,12\,y^2 - 8 = \cancel{8(y^2-1)}\,4(3y^2-2)$$

$(0,0)$ $\begin{vmatrix} -8 & 8 \\ 8 & -8 \end{vmatrix} = 0 \rightarrow$ 判定不能.

$(\pm 2, \mp 2)$ $\begin{vmatrix} 4\cdot 10 & -8 \\ -8 & 4\cdot 10 \end{vmatrix} = 8^2 \begin{vmatrix} 5 & -1 \\ -1 & 5 \end{vmatrix}$

$$\left.\begin{array}{l} = 8^2 \cdot 24 > 0 \\ z_{xx} = 4\cdot 10 > 0 \end{array}\right\} \rightarrow \text{極小}$$

A: 2次正方行列, 対称 $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$

$(A\vec{x}, \vec{x})$ 非負定値 非正定値

$\Longleftrightarrow$ $(A\vec{x}, \vec{x}) \geqq 0$

$\leqq$.

$\Longleftrightarrow$ $a, b \geqq 0$, $\det(A) \geqq 0$

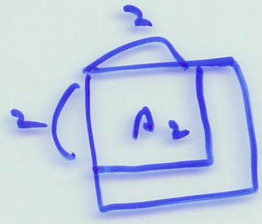
$\leqq 0$ $\geqq 0$

$(0,0)$の近くで $\ni (x,y)$ $(H(f)(x,y)\vec{\xi}, \vec{\xi})$ は 非正定値

~~[illegible]~~ $\cancel{f'(a) = 0,\ f''(a) \geqq 0.}$

3変数

A: 実対称行列, 3次.



$(A\vec{x},\vec{x})$ 正定値.

$\Longleftrightarrow$ $a_{11}>0$, $\det(A_2)>0$, $\det(A)>0$

$\Rightarrow$ (は相対 A) は easy.

$\det(A)>0$. が必要

P: 直交行列.

$${}^tPAP=\begin{pmatrix}\alpha & & \\ & \beta & \\ & & \gamma\end{pmatrix}$$

${}^tP$: 直交

$$(A\vec{x},\vec{x})=({}^tPA\boxed{P\,{}^tP}\vec{x},\ {}^tP\vec{x})$$

$$P\,{}^tP = I_3$$

$${}^tP\vec{x}=\vec{\xi}$$

$$=\left(\begin{pmatrix}\alpha & & \\ & \beta & \\ & & \gamma\end{pmatrix}\vec{\xi},\ \vec{\xi}\right)$$

$$=\alpha\xi_1^2+\beta\xi_2^2+\gamma\xi_3^2>0\ (\vec{\xi}\neq\vec{0})$$

$(A\vec{x},\vec{x})>0$ $(\vec{x}\neq\vec{0})$ $\Longleftrightarrow$ $\alpha,\beta,\gamma>0$.

$\Updownarrow$

$\vec{\xi}\neq\vec{0}$

$\det(A)=\det({}^tPAP)=\alpha\beta\gamma>0$ が必要

$\det({}^tPAP) = |{}^tP|\cdot|A|\,|P|$

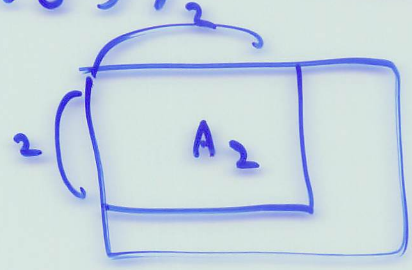
$$\boxed{\det(A)>0\ \text{が必要.}}$$

B: 2次 実対称行列.

$(B\vec{y},\vec{y})$ 正定値 $\Longleftarrow$ $b_{11}>0$, $\det(B)>0$.

$$\left(A\begin{pmatrix}x_1\\x_2\\0\end{pmatrix}, \begin{pmatrix}x_1\\x_2\\0\end{pmatrix}\right) = (A_2\vec{y}, \vec{y}) > 0$$

$\vec{y} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ $\qquad (\vec{y} \neq \vec{0})$



$$\rightsquigarrow \quad a_{11} > 0, \ \det(A) > 0$$

証 $\quad a_{11} > 0, \ \det(A_2) > 0, \ \det(A)$

$\Rightarrow (A\vec{x}, \vec{x})$ 正定値

$(A\vec{v}, \vec{v}) > 0 \quad (\vec{v} \neq \vec{0})$

$\vec{v} = \boxed{\left(\begin{array}{cc|c}1 & 0 & t_1\\ 0 & 1 & t_2\\ \hline 0 & 0 & 1\end{array}\right)} \vec{w}$ と変数変換する。 $= T$ 正則

$\begin{pmatrix}t_1\\t_2\end{pmatrix} = \vec{t}$

t_1, t_2 は後に定める。

$$(A\vec{v}, \vec{v}) = (AT\vec{w}, T\vec{w})$$

$$= ({}^tTAT\vec{w}, \vec{w})$$

$${}^tTAT = \left(\begin{array}{c|c} A_2 & A_2\vec{t} + \vec{a} \\ \hline & \delta \end{array}\right) = \left(\begin{array}{c|c} A_2 & \begin{matrix}0\\0\end{matrix} \\ \hline 0\ 0 & \delta \end{array}\right)$$

$$A = \left(\begin{array}{c|c} A_2 & \vec{a} \\ \hline {}^t\vec{a} & a_{33} \end{array}\right) \qquad {}^t(A_2\vec{t} + \vec{a})$$

$$\delta = {}^t\vec{t} A_2 \vec{t} + 2\,{}^t\vec{t}\vec{a} + a_{33}$$

$\det(A_2) > 0$ $\neq 0$ $\quad \rightsquigarrow \vec{t} = -A_2^{-1}\vec{a}$

$$\rightsquigarrow \underline{A_2\vec{t} + \vec{a} = \vec{0}}$$

$$(A\vec{v}, \vec{v}) = ({}^tT\,A\,T\vec{w}, \vec{w})$$

$$= (A_2\vec{y}, \vec{y}) + \delta w_3^2$$

$$\vec{y} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\delta = {}^t\vec{t}\,A_2\,\vec{t} + 2\,{}^t\vec{t}\,\vec{a} + a_{33}$$

$$\boxed{\delta > 0} \qquad {}^tT\,A\,T = \left(\begin{array}{c|c} A_2 & \\ \hline & \delta \end{array}\right)$$

$$\det({}^tT\,A\,T) = \det(A_2)\cdot\delta$$

$\det(A_2)$ $\vee$ 0 仮定

$$\det({}^tTAT) = \underbrace{\boxed{\det({}^tT)}}_{1}\,\underbrace{\boxed{\det(A)}}_{\vee\, 0}\,\underbrace{\boxed{\det(T)}}_{1}$$

$$T = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \qquad \det(T) = 1$$

仮定

$$(A\vec{v}, \vec{v}) = (A_2\vec{y}, \vec{y}) + \delta w_3^2 \geqq 0$$

$$\delta \;\vee\; 0$$

$$\vec{y} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

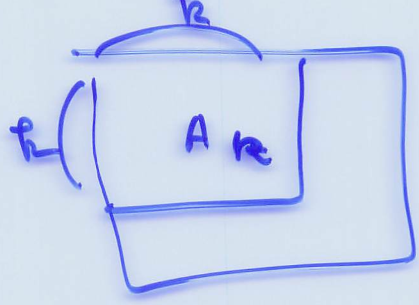
$$\vec{w} \neq \vec{0} \quad \begin{cases} \vec{y} \neq \vec{0} \longrightarrow (A_2\vec{y}, \vec{y}) > 0 \\ w_3 \neq 0 \qquad \delta w_3^2 > 0 \end{cases}$$

A : n次対称行列.

$(A\vec{x}, \vec{x}) > 0$ $<$

$(\vec{x} \neq \vec{0})$

$\Updownarrow$



主座小行列.

$a_{11} > 0$, $\det(A_2) > 0$, $\cdots$, $\det(A) > 0$

$<$ $\qquad$ $(-1)^n$

$\det(A_3) < 0$

$w = f(x, y, z)$

$f_x(a, b, c) = f_y(a, b, c) = f_z(a, b, c) = 0$

とする.

$$H = H(f)(a, b, c) = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

$(H(f)(a, b, c)\vec{\xi}, \vec{\xi})$ 正定値 $\Longrightarrow$ (a, b, c) 極小.

負 $\qquad$ 大

演習

$w = x^3 + y^3 + z^3 - 3xz - 3y^2$

の停留点を求める.