(2) $z = x^3 + y^3 - 3xy$

停留点 $z_x = z_y = 0$

定理　$z = f(x, y)$ が (a, b) で極値

$\Longrightarrow f_x(a,b) = f_y(a,b) = 0$

$$z_x = 3x^2 + 0 - 3y \cdot 1 = 3(x^2 - y) = 0$$
$$z_y = 0 + 3y^2 - 3x \cdot 1 = 3(y^2 - x) = 0$$

$\longrightarrow$ $y = x^2$ …①

かつ

$x = y^2$ …②

① と ② により　$x = x^4$

$$x^4 - x = x(x^3 - 1)$$
$$= x(x-1)\boxed{(x^2 + x + 1)}$$

1°

$x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4}$

よって　$x = 0$ または $x = 1$

① により　$y = 0$　　$y = 1$

$(x, y) = (0, 0), (1, 1)$ が停留点

定理 (a,b) が f の停留点とする。

(i) $f_{xx}(a,b) > 0$ ($<$), $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$ at (a,b) 　固有値 正, 正

H が定める 2 次形式は 正定値. (負) 固有値 負, 負.

$\Longrightarrow$ (a,b) で 極小. (大)

(ii) $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} < 0$ $\rightsquigarrow$ 極大でも極小でもない.

$\overset{\shortparallel}{H}$ 　H には 正の固有値と負の固有値あり.

$z = x^3 + y^3 - 3xy$

$z_x = 3x^2 - 3y$, $z_y = 3y^2 - 3x$.

$z_{xx} = 6x$, $z_{xy} = -3$, $z_{yy} = 6y$.
$\quad\quad\quad\quad\;\; \shortparallel$
$\quad\quad\quad\quad\; z_{yx}$

$$|H_2| = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} = 3^2 \begin{vmatrix} 2x & -1 \\ -1 & 2y \end{vmatrix}$$
$$= 9(4xy - 1)$$

(i) $(0,0)$ 　$|H_2| = 9(-1) = -9 < 0$ 　極大でも極小でもない.

(ii) $(1,1)$ 　$|H_2| = 9(4-1) = 27 > 0$

$z_{xx} = 6 > 0$ 　$\longrightarrow$ 極小

H: 2次対称行列.

固有値 α, β $\quad H = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$

P: 直交行列.

$$ {}^tPHP = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} $$

$$ \det({}^tPHP) = \det\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \alpha\beta $$

$$ \det({}^tPHP) = \boxed{\det({}^tP)\det(H)\det(P)} $$

$$ \det({}^tP)\det(P) = \det({}^tPP) = \det(I_2) = 1 $$

$$ = \det(H) $$

$$ \det(H) = \alpha\beta. $$

$\det(H) < 0 \iff \alpha, \beta$ の符号が違う.

前回 $\alpha, \beta > 0 \iff a > 0, \ |H| > 0$

$\qquad\quad <\qquad\qquad <$

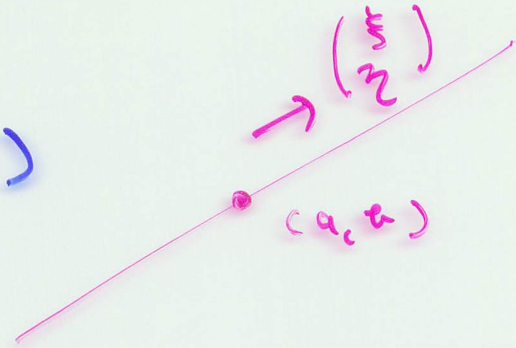
$f_x(a,b) = f_y(a,b) = 0$

(ii) $\begin{vmatrix} f_{xx}(a,b) & f_{xy}(\ \ ) \\ f_{yx}(\ \ ) & f_{yy}(\ \ ) \end{vmatrix} < 0$

$\rightsquigarrow$ (a,b)は極大でも極小でもない.

$F(t)$: $\quad F'(a) = 0, \ F''(a) > 0$ ($<$)

$\rightsquigarrow$ aは極小. (大)



$F(t) = f(a+\xi t,\ b+\eta t)$

とおく.

$$\begin{aligned}
&F(t) = f(x(t),\ y(t)) \\
&F'(t) = f_x(x(t), y(t))\,x'(t) \\
&\qquad + f_y(\quad)\,y'(t)
\end{aligned}$$

$F'(t) = f_x \cdot \xi + f_y \cdot \eta$

$F'(0) = \underbrace{\boxed{f_x(a,b)}}_{=0}\xi + \underbrace{\boxed{f_y(a,b)}}_{=0}\eta$

$= 0$

$F''(t) = \xi\left(f_{xx}\cdot\xi + f_{xy}\cdot\eta\right) + \eta\left(f_{yx}\cdot\xi + f_{yy}\eta\right)$

$= f_{xx}\xi^2 + 2f_{xy}\xi\eta + f_{yy}\eta^2$

$f_{xy} = f_{yx}$ $\quad H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$

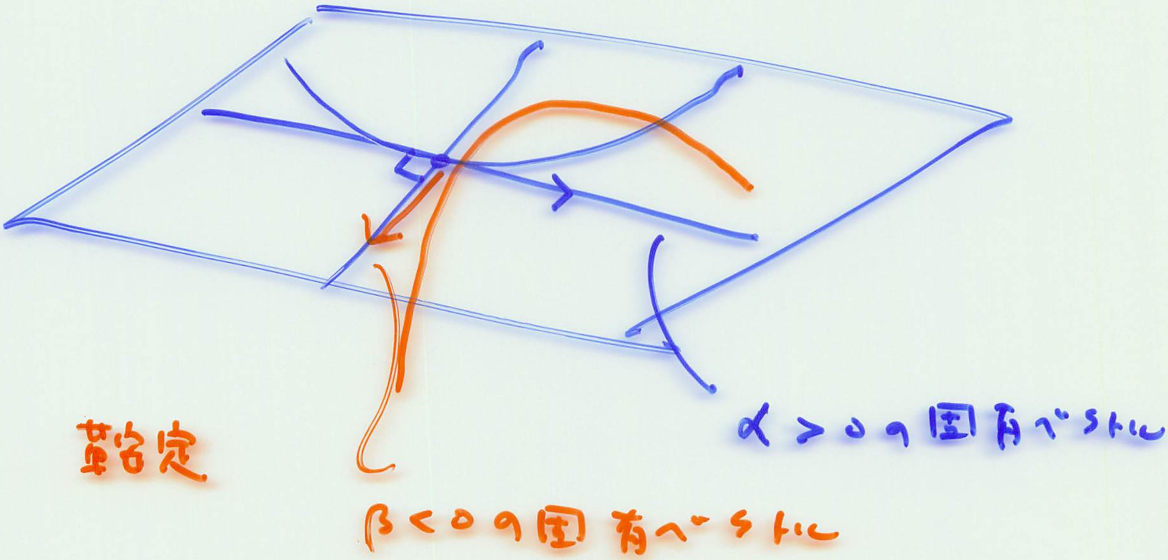
$F''(0) = \left(H(a,b)\begin{pmatrix}\xi\\ \eta\end{pmatrix},\ \begin{pmatrix}\xi\\ \eta\end{pmatrix}\right)$

(i) $\begin{pmatrix}\xi\\ \eta\end{pmatrix}$ が $H(a,b)$ の α の固有ベクトル $\alpha > 0$

$H\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \underset{\beta}{\alpha}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \rightsquigarrow F''(0) = \left(\alpha\begin{pmatrix}\xi\\ \eta\end{pmatrix},\ \begin{pmatrix}\xi\\ \eta\end{pmatrix}\right)$

$= \overset{\beta}{\alpha}(\xi^2+\eta^2) \underset{<}{>} 0$

(ii) $\begin{pmatrix}\xi\\ \eta\end{pmatrix}$ が ―― $\beta < 0$ ――

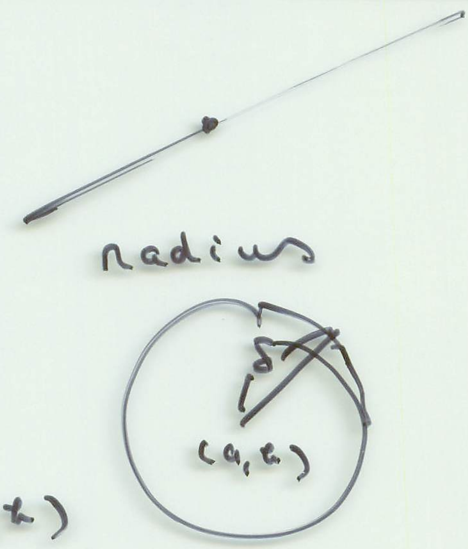


(i) の証明. Taylor の定理

$$F(t) = F(a) + F'(t)(t-a) + \frac{F''(\overset{c}{\downarrow}c)}{2!}(t-a)^2$$

をみたす c が a と t の間にあり.

連続性　f(x, y) が連続.



$f(a, b) > 0$ とする.

⟶ 小さな $\delta > 0$ があって

$f(x, y) > 0$

$(x, y) \in B_\delta(a, b)$

中心 (a, b) 半径 $\delta > 0$ の開円盤.

$$f_x(a,b) = f_y(a,b) = 0$$

1) $\begin{vmatrix} f_{xx}(a,b) & f_{xy}(\ ) \\ f_{yx}(a,b) & f_{yy} \end{vmatrix} > 0$

2) $f_{xx}(a,b) > 0$

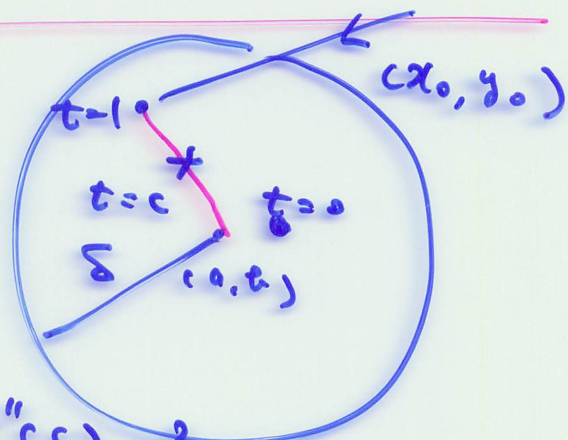
仮定 f, f_x, f_y, f_{xx} etc : 連続.

$B_\delta(a,b) \ni (x,y)$
$H(x,y)$ は正定値.

$f_{xx}(x,y) > 0$ $\quad (x,y) \in B_\delta(a,b)$

$\begin{vmatrix} & \end{vmatrix}(x,y) > 0$ $\quad (x,y) \in B_\delta(a,b)$

$\xi = x_0 - a$,
$\eta = y_0 - b$



$$F(t) = f(a+\xi t, b+\eta t)$$

$$F(1) - F(0) = \overbrace{F'(0)}^{=0}\cdot 1 + \frac{F''(c)}{2}\cdot 1^2$$

$\cancel{f(a,b) = }$

$$f(a+(x_0-a), b+(y_0-b)) - f(a,b)$$
$$= \cancel{f(\ )}\ f(x_0, y_0) - f(a,b)$$

$$F'(0) = \underbrace{\boxed{f_x(a,b)}}_{=0}\xi + \underbrace{\boxed{f_y(a,b)}}_{=0}\eta$$

(a,b)で極小.

$\boxed{f}$ $f(x_0, y_0) - f(a,b)$

$$= \frac{1}{2}\left(H(a+\xi c, b+\eta c)\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \begin{pmatrix}\xi\\ \eta\end{pmatrix}\right) > 0.$$

$$\rightsquigarrow f(x_0, y_0) > f(a,b)$$

I

6

$z = x^4 + y^4 - 4(x-y)^2$

の極値を求めよ.

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$(x^2 - xy + y^2) = \left(x - \frac{1}{2}y\right)^2 + \frac{3}{4}y^2$$

$$x^2 - xy + y^2 = 0 \iff x = \frac{1}{2}y \text{ かつ } y = 0$$

$$\iff x = y = 0$$