$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 7 & -4 \\ 2 & -4 & 1 \end{pmatrix} \longleftarrow \text{実対称}$$

$$\Phi_A(\lambda) := |\lambda I_3 - A| = (\lambda+1)^2(\lambda-1)$$

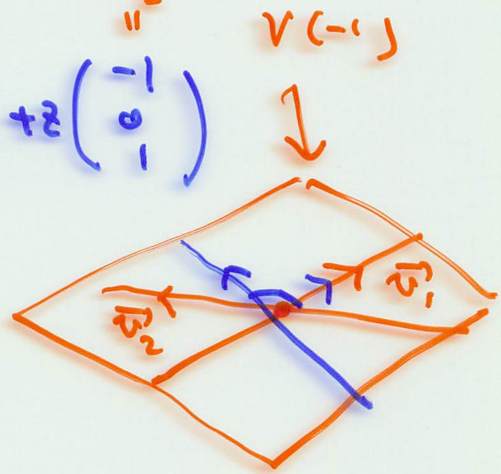
$$V(-1) = \operatorname{Ker}(-I_3 - A)$$

$$-I_3 - A = \begin{pmatrix} -2 & 4 & -2 \\ 4 & -8 & 4 \\ -2 & 4 & -2 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & -2 & 1 \\ \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}} \end{pmatrix}$$

$$0x + 0y + 0z = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V(-1) \iff x - 2y + z = 0$$

$$= \begin{pmatrix} 2y - z \\ y \\ z \end{pmatrix} = y \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_1} + z \overbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}^{\vec{v}_2}$$



$$(\vec{v}_1, \vec{v}_2) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -2 \neq 0$$

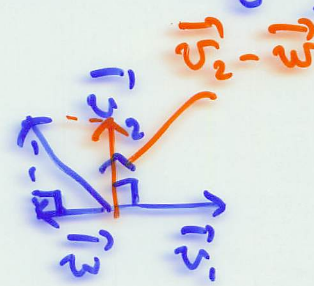
$$\vec{p}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{\sqrt{5}} \vec{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$\vec{v}_2$ の $\vec{v}_1$ 方向への正射影を $\vec{w}$

$$\vec{w} = \frac{(\vec{v}_1, \vec{v}_2)}{\|\vec{v}_1\|^2} \vec{v}_1 = \frac{-2}{5} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ -\frac{2}{5} \\ 0 \end{pmatrix}$$

$$\vec{v}_2 - \vec{w} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -\frac{4}{5} \\ -\frac{2}{5} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \\ 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$$



$$\vec{p}_2 = \frac{1}{\sqrt{30}} \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$$

$\vec{c}$ の $\vec{a} \neq \vec{0}$ 方向への正射影 $\vec{w}$

$$\vec{w} = \frac{(\vec{a}, \vec{c})}{\|\vec{a}\|^2} \vec{a}$$

$V(-1)$ $\vec{p}_1, \vec{p}_2$

z : pivot なし

$V(11)$ $11 I_3 - A \to \cdots \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

pivot $= 1$

pivot の上 0

$$\begin{cases} x \quad - z = 0 \\ \quad y + 2z = 0 \end{cases}$$

$z = \alpha$ とおく.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ -2\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

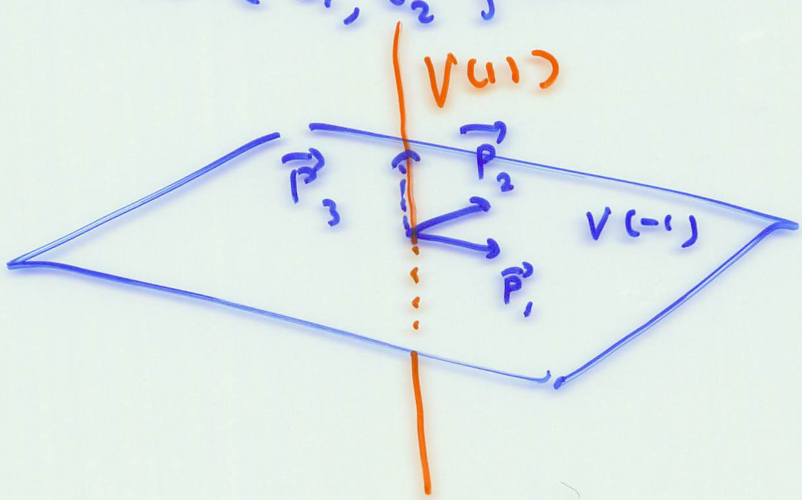
$$\vec{p}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

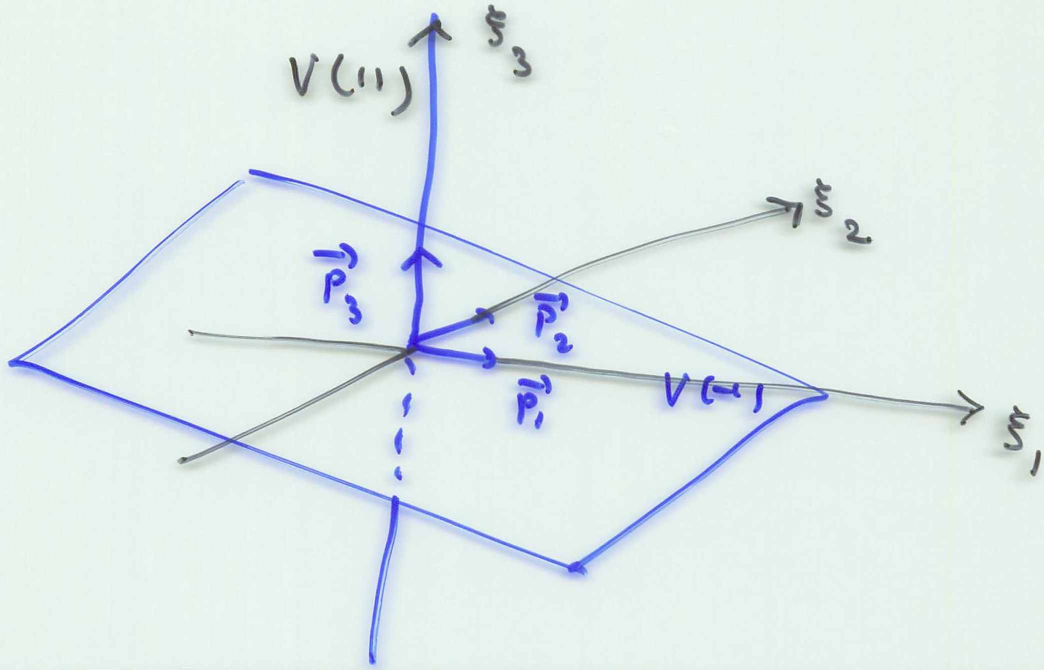
何故? $(\vec{p}_1, \vec{p}_3) = (\vec{p}_2, \vec{p}_3) = 0$

定理 A : 実対称行列, α, β : A の固有値 $\alpha \neq \beta$.

$$A\vec{v}_1 = \alpha \vec{v}_1, \quad A\vec{v}_2 = \beta \vec{v}_2$$

$$\rightsquigarrow (\vec{v}_1, \vec{v}_2) = 0$$





$$\vec{x} = (\vec{p}_1\ \vec{p}_2\ \vec{p}_3)\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \xi_1\vec{p}_1 + \xi_2\vec{p}_2 + \xi_3\vec{p}_3 \rightsquigarrow \boxed{\begin{array}{l}\vec{x} = P\vec{\xi}\\ {}^tP\vec{x} = \vec{\xi}\end{array}}$$

$$(A\vec{x}, \vec{x}) = ({}^tP A \overbrace{P\,{}^tP}^{I_3} \vec{x}, {}^tP\vec{x})$$

${}^tP$: 直交

P : 直交 $\quad {}^tP P = P\,{}^tP = I_3 \rightsquigarrow {}^tP({}^t({}^tP)) = {}^t({}^tP)\,{}^tP = I_3$

${}^t({}^tP) = P$

$$(P\vec{x}, P\vec{y}) = (\vec{x}, \vec{y})$$

$${}^tPAP = \begin{pmatrix}-1 & & \\ & -1 & \\ & & 11\end{pmatrix}, \quad {}^tP\vec{x} = \vec{\xi}$$

$$= \left(\begin{pmatrix}-1 & & \\ & -1 & \\ & & 11\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}, \begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}\right)$$

$$= -\xi_1^2 - \xi_2^2 + 11\xi_3^2$$

A：実対称行列 $(n=3)$

① Aの固有値は全て実

$$\Phi_A(\lambda) = (\lambda-\alpha_1)(\lambda-\alpha_2)\cdots(\lambda-\alpha_n)$$

$$\alpha_1, \cdots, \alpha_n \in \mathbb{R}.$$

② 直交行列 Pが存在し

$$AP = P\begin{pmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_n\end{pmatrix} \qquad {}^tPP = I_n$$

$$A = P\begin{pmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_n\end{pmatrix}{}^tP \qquad P^{-1} = {}^tP.$$

$$AP = P\begin{pmatrix}-1 & & \\ & -1 & \\ & & 11\end{pmatrix} \qquad P：直交.$$

$$\vec{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \begin{pmatrix}x\\y\\z\end{pmatrix}$$

<u>2次形式</u>

$$(A\vec{x}, \vec{x})$$

$$A = \begin{pmatrix}1 & -4 & 2\\ -4 & 7 & -4\\ 2 & -4 & 1\end{pmatrix}$$

$$\left(A\begin{pmatrix}x\\y\\z\end{pmatrix}, \begin{pmatrix}x\\y\\z\end{pmatrix}\right) = \begin{pmatrix}x-4y+2z\\ -4x+7y-4z\\ 2x-4y+z\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$= x^2 + 7y^2 + z^2$$

$$-8xy - 8yz + 4xz.$$

$w = f(x, y, z)$

定理 w が $(x, y, z) = (a, b, c)$ で 極小.
or 極大

$\Longrightarrow$ (#) $\dfrac{\partial f}{\partial x}(a, b, c) = \dfrac{\partial f}{\partial y}(a, b, c) = \dfrac{\partial f}{\partial z}(a, b, c) = 0$

i.e. (a, b, c) は f の停留点.

ido est

$F(x) = f(x, a, b)$ が $x = a$ で 極小(大)

$\leadsto$ $F'(a) = f_x(a, b, c) = 0.$

定理 (#) を仮定する. f は C^2 級 を仮定する

$$H(f) = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

$f_{xy} = f_{yx}$

実対称

ヘッセ行列.

$(H(f)(a, b, c)\vec{v}, \vec{v}) \overset{<}{>} 0 \quad (\vec{v} \neq \vec{0})$

$\longrightarrow$ (a, b, c) で 極小.
大.

このうち 証明.

A: 実対称.

$(A\vec{x}, \vec{x}) > 0 \quad (\vec{x} \neq 0)$: 2次形式は 正定値.

< 0 負.

$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ 実対称.

$$\begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = (\lambda - a)(\lambda - b) - c^2$$
$$= \lambda^2 - (a+b)\lambda + ab - c^2 = 0$$

$$D = (a+b)^2 - 4(ab - c^2)$$
$$= (a-b)^2 + 4c^2 \geqq 0$$

$D = 0 \iff a = b,\ c = 0$

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI_2.$$

固有値 α, β

<u>定理</u> $(A\vec{x}, \vec{x})$: 正定値. (負定値)

$\iff \alpha, \beta > 0 \iff a > 0,\ |A| > 0$

$\qquad\quad < 0 \qquad\qquad a < 0,\ |A| > 0.$

$(A\vec{x}, \vec{x}) < 0 \quad (\vec{x} \neq \vec{0})$

$AP = P\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ P : 直交

$$(A\vec{x}, \vec{x}) = ({}^tP A \underbrace{P\,{}^tP}_{= I_2} \vec{x},\ {}^tP\vec{x})$$

$$= \left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}\right)$$

${}^tP\vec{x} = \vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$

$$= \alpha\xi_1^2 + \beta\xi_2^2$$

$\boxed{\vec{x} \neq \vec{0} \iff \vec{\xi} \neq \vec{0}}$

${}^tP$: 正則

$(A\vec{x}, \vec{x}) > 0 \quad (\vec{x} \neq \vec{0})$

$\Updownarrow$

$\alpha\xi_1^2 + \beta\xi_2^2 > 0 \quad \left(\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \neq \vec{0}\right)$

★ $\alpha \xi_1^2 + \beta \xi_2^2 > 0 \quad \left(\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \neq \vec{0} \right)$

$\Updownarrow$

$\alpha, \beta > 0$

(i) $\alpha, \beta > 0$ とする.

$\alpha \xi_1^2 + \beta \xi_2^2 \geqq 0$ → $> 0.$ → 正定値 ★

$\xi_1 \neq 0 \rightsquigarrow \alpha \xi_1^2 > 0$

$\xi_2 \neq 0 \rightsquigarrow \beta \xi_2^2 > 0$

(ii) ★ $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightsquigarrow \alpha > 0$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightsquigarrow \beta > 0$

$\alpha, \beta > 0 \iff \alpha + \beta > 0, \ \alpha\beta > 0$

$\lambda^2 - (\alpha+\beta)\lambda + \alpha\beta = \lambda^2 - (a+b)\lambda + |A|$

$\boxed{\alpha\beta = |A|, \quad \alpha+\beta = a+b.}$

$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$

(i) $\alpha + \beta > 0, \ \alpha\beta > 0 \rightsquigarrow |A| > 0$

$\Downarrow$

$a + b$

$\alpha\beta = |A| = ab - c^2 \leqq ab$

$\rightsquigarrow ab > 0, \ a + b > 0$

$\rightsquigarrow a, b > 0 \rightarrow a > 0$

(ii) $a > 0, \ |A| > 0$

$|A| = \alpha\beta$, $|A| = ab - c^2$

$ab - c^2 > 0$

$ab > c^2 \geqq 0$

$\rightsquigarrow ab > 0$

$a > 0, \ ab > 0 \rightsquigarrow b > 0$

$\alpha + \beta = a + b > 0.$

$\alpha\beta > 0, \ \alpha + \beta > 0.$

定理 $f_x(a,b) = f_y(a,b) = 0$

$f_{xx}(a,b) > 0$, $\begin{vmatrix} f_{xx}(a,b) & f_{xy}(\ ) \\ f_{yx} & f_{yy} \end{vmatrix} > 0$

$<$

$\Longrightarrow$ (a,b) 極小, $(f_{yx}) = (f_y)_x$

大.

(i) $f(x,y) = x^3 + y^3 - 3xy$

の停留点を求める.

(ii) 極大・極小の判定を行なう.